

# ON ORDER STRUCTURE OF THE SET OF ONE-POINT TYCHONOFF EXTENSIONS OF A LOCALLY COMPACT SPACE

M.R. KOUSHESH

**ABSTRACT.** If a Tychonoff space  $X$  is dense in a Tychonoff space  $Y$ , then  $Y$  is called a *Tychonoff extension* of  $X$ . Two Tychonoff extensions  $Y_1$  and  $Y_2$  of  $X$  are said to be *equivalent*, if there exists a homeomorphism  $f : Y_1 \rightarrow Y_2$  which keeps  $X$  pointwise fixed. This defines an equivalence relation on the class of all Tychonoff extensions of  $X$ . We identify those extensions of  $X$  which belong to the same equivalence classes. For two Tychonoff extensions  $Y_1$  and  $Y_2$  of  $X$ , we write  $Y_2 \leq Y_1$ , if there exists a continuous function  $f : Y_1 \rightarrow Y_2$  which keeps  $X$  pointwise fixed. This is a partial order on the set of all (equivalence classes of) Tychonoff extensions of  $X$ . If a Tychonoff extension  $Y$  of  $X$  is such that  $Y \setminus X$  is a singleton, then  $Y$  is called a *one-point extension* of  $X$ . Let  $\mathcal{T}(X)$  denote the set of all one-point extensions of  $X$ . Our purpose is to study the order structure of the partially ordered set  $(\mathcal{T}(X), \leq)$ . For a locally compact space  $X$ , we define an order-anti-isomorphism from  $\mathcal{T}(X)$  onto the set of all non-empty closed subsets of  $\beta X \setminus X$ . We consider various sets of one-point extensions, including the set of all one-point locally compact extensions of  $X$ , the set of all one-point Lindelöf extensions of  $X$ , the set of all one-point pseudocompact extensions of  $X$ , and the set of all one-point Čech-complete extensions of  $X$ , among others. We study how these sets of one-point extensions are related, and investigate the relation between their order structure, and the topology of subspaces of  $\beta X \setminus X$ . We find some lower bounds for cardinalities of some of these sets of one-point extensions, and in a concluding section, we show how some of our results may be applied to obtain relations between the order structure of certain subfamilies of ideals of  $C^*(X)$ , partially ordered with inclusion, and the topology of subspaces of  $\beta X \setminus X$ . We leave some problems open.

## 1. INTRODUCTION

Let  $X$  be a Tychonoff space. If a Tychonoff space  $Y$  contains  $X$  as a dense subspace, we call  $Y$  a *Tychonoff extension* of  $X$ . Two Tychonoff extensions  $Y_1$  and  $Y_2$  of  $X$  are said to be *equivalent*, if there exists a homeomorphism  $f : Y_1 \rightarrow Y_2$  which keeps  $X$  pointwise fixed. This indeed defines an equivalence relation which splits the class of all Tychonoff extensions of  $X$  into equivalence classes. We identify the equivalence classes with individuals whenever no confusion arises. For two Tychonoff extensions  $Y_1$  and  $Y_2$  of  $X$ , we write  $Y_2 \leq Y_1$ , if there exists a continuous function  $f : Y_1 \rightarrow Y_2$  which keeps  $X$  pointwise fixed. This defines a partial order on the set of all (equivalence classes of) Tychonoff extensions of  $X$ . A detailed study of this partial order can be found in Section 4.1 of [12]. If an extension  $Y$  of  $X$  is such that  $Y \setminus X$  consists of a single element, then  $Y$  is called a *one-point extension*

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2000 *Mathematics Subject Classification.* 54D35, 54D40, 54A25.

*Key words and phrases.* One-point extension; One-point compactification; Stone-Čech compactification;  $\beta X \setminus X$ ; Rings of continuous functions.

of  $X$ . Let  $\mathcal{T}(X)$  denote the set of all one-point extensions of  $X$ . Our purpose here is to study the order structure of the partially ordered set  $(\mathcal{T}(X), \leq)$ .

First we define some notations and terminologies we will use. For a Tychonoff space  $X$ , we let  $\beta X$  and  $\nu X$  denote the Stone-Ćech compactification of  $X$  and the Hewitt realcompactification of  $X$ , respectively. For a subset  $A$  of  $X$ , we let  $A^* = (\text{cl}_{\beta X} A) \setminus X$ . In particular,  $X^* = \beta X \setminus X$ . For a space  $X$ , we denote the set of all closed subset of  $X$ , the set of all zero-sets of  $X$ , and the set of all clopen (open-closed) subsets of  $X$ , by  $\mathcal{C}(X)$ ,  $\mathcal{Z}(X)$  and  $\mathcal{B}(X)$ , respectively. A space  $X$  is called *zero-dimensional*, if the set  $\mathcal{B}(X)$  is an open base for  $X$ . For two spaces  $X$  and  $Y$ ,  $C(X, Y)$  denotes the set of all continuous functions from  $X$  to  $Y$ . The letters  $\mathbf{I}$  and  $\mathbf{R}$ , denote the closed unit interval and the real line, respectively. We also let  $C(X) = C(X, \mathbf{R})$ , and we denote by  $C^*(X)$  the set of all bounded elements of  $C(X)$ .

We denote by  $\omega$  the first countably infinite ordinal number, and we denote by  $\aleph_0$  the cardinality of  $\omega$ . By [CH] and [MA] we mean the Continuum Hypothesis and the Martin's Axiom, and whenever they appear at the beginning of the statement of a theorem, indicate that they have been assumed in the proof of that theorem.

In a partially ordered set  $P$ , the two symbols  $\bigvee$  and  $\bigwedge$  are used to denote the least upper bound and the greatest lower bound (provided that they exist) respectively. The elements  $\bigvee P$  and  $\bigwedge P$  are called the *maximum* and the *minimum* of  $P$ , respectively. An element  $p \in P$  is called a *maximal* (*minimal*, respectively) element of  $P$ , provided that for every  $x \in P$ , if  $x \geq p$  ( $x \leq p$ , respectively) then  $x = p$ . If  $P$  and  $Q$  are partially ordered sets, a function  $f : P \rightarrow Q$  is called an *order-homomorphism* (*order-anti-homomorphism*, respectively) if  $f(a) \leq f(b)$  ( $f(a) \geq f(b)$ , respectively) whenever  $a \leq b$ . The function  $f$  is called an *order-isomorphism* (*order-anti-isomorphism*, respectively) if it is moreover bijective, and  $f^{-1} : Q \rightarrow P$  also is an order-homomorphism (order-anti-homomorphism, respectively). The partially ordered sets  $P$  and  $Q$  are called *order-isomorphic* (*order-anti-isomorphic*, respectively) if there is an order-isomorphism (order-anti-isomorphism, respectively) between them.

For terms and notations not defined here we follow the standard text of [4]. In particular, compact and paracompact spaces are assumed to be Hausdorff, and perfect mappings are assumed to be continuous. By a neighborhood of a point  $x$  in a space  $X$ , we mean a subset of  $X$  which contains an open subset containing  $x$ .

In 1924, P. Alexandroff proved that a locally compact non-compact space  $X$  has a one-point extension which is compact. This is now known as the Alexandroff compactification of  $X$ , or the one-point compactification of  $X$ . Since then, one-point extensions have been studied extensively by various authors (for some results as well as some bibliographies on the subject see [10] and [11]). The majority of these works, however, deals with conditions under which if a space locally possesses a topological property  $\mathcal{P}$ , then it has a one-point extension which has  $\mathcal{P}$ . Recently, M. Henriksen, L. Janos and R.G. Woods have studied the partially ordered set of all one-point metrizable extensions of a locally compact metrizable space, by relating it to the topology of subspaces of  $X^*$ . Here is a brief summary of the method they applied. Let  $X$  be a (non-compact) metrizable space. We call a sequence  $\{U_n\}_{n < \omega}$  of non-empty open subsets of  $X$  an *extension trace* in  $X$ , if for each  $n < \omega$  we have  $\text{cl}_X U_{n+1} \subseteq U_n$  and  $\bigcap_{n < \omega} U_n = \emptyset$ . To every one-point metrizable extension  $Y = X \cup \{p\}$  of  $X$ , we can correspond an extension trace of

$X$ , namely,  $U_n = B(p, 1/n) \cap X$ . Conversely, if  $\{U_n\}_{n < \omega}$  is an extension trace in  $X$ , let  $Y = X \cup \{p\}$ , where  $p \notin X$ , and define a topology on  $Y$  consisting of sets of the form  $V \cup \{p\}$ , where  $V$  is open in  $X$  and is such that  $V \supseteq U_n$ , for some  $n < \omega$ . By Theorem 2 of [1] (or Theorem 3.4 of [2]) the space  $Y$  thus defined is metrizable, and therefore it is a one-point metrizable extension of  $X$ . It may happen, however, that different extension traces in  $X$  give rise to the same one-point extension of  $X$ . To fix this problem, we define an equivalence relation on the set of all extension traces of  $X$ . Two extension traces  $\{U_n\}_{n < \omega}$  and  $\{V_n\}_{n < \omega}$  of  $X$  are said to be *equivalent*, if for each  $n < \omega$ , there exist  $k_n, l_n < \omega$ , such that  $U_n \supseteq V_{k_n}$  and  $V_n \supseteq U_{l_n}$ . This makes a (one-one) correspondence between the set of all equivalence classes of extension traces of  $X$ , and the set of all one-point metrizable extensions  $\mathcal{E}(X)$  of  $X$ . Using this, we can define a function  $\lambda : \mathcal{E}(X) \rightarrow \mathcal{Z}(X^*)$  by  $\lambda(Y) = \bigcap_{n < \omega} U_n^*$ , where  $\{U_n\}_{n < \omega}$  is an extension trace in  $X$  which generates  $Y$ . It is proved in [7] that the function  $\lambda$  is well-defined, and it is an order-anti-isomorphism onto its image (in the case when  $X$  is moreover separable, it is proved in [7] that  $\lambda$  maps  $\mathcal{E}(X)$  onto  $\mathcal{Z}(X^*) \setminus \{\emptyset\}$ ). Using the function  $\lambda$ , and the fact that the topology of any compact space determines and is determined by the order structure of the set of its all zero-sets, the authors of [7] have studied the order structure of sets of one-point metrizable extensions of a locally compact metrizable space  $X$ , by relating it to the topology of certain subspaces of  $X^*$ . Motivated by the results of [7] and the author's earlier work [8] (which is in fact a continuation of the work of Henriksen, Janos and Woods, in which we generalized most of the results of [7] from the separable case to the non-separable case) we define an order-anti-isomorphism  $\mu : \mathcal{T}(X) \rightarrow \mathcal{C}(X^*) \setminus \{\emptyset\}$ . Using the mapping  $\mu$  we will be able to relate the topology of certain subspaces of  $X^*$  to the order structure of various sets of one-point extensions of  $X$ . These sets of one-point extensions include, the set of all one-point locally compact extensions of  $X$ , the set of all one-point Lindelöf extensions of  $X$ , the set of all one-point pseudocompact extensions of  $X$ , and the set of all one-point Čech-complete extensions of  $X$ , among others.

In Section 2, we define certain sets of one-point extensions. We then establish the order-isomorphism  $\mu$ , mentioned above, from the set of all one-point extensions of the locally compact space  $X$  onto the set of all non-empty closed subsets of  $X^*$ . We find the image under  $\mu$  of some of the sets of one-point extensions introduced before, and as a result, we show that the order structure of some of them determines and is determined by the topology of the space  $X^*$ . In Section 3, we obtain some results relating the order structure of certain sets of one-point extensions of  $X$  and the topology of subspaces of  $X^*$ , under the extra assumption of paracompactness of  $X$ . Section 4 deals with the order theoretic relations between various sets of one-point extensions the space  $X$ . In Section 5, we find sufficient conditions that some of the sets of one-point extensions admit maximal or minimal elements. In section 6, we find a lower bound for cardinalities of two of the sets of one-point extensions introduced before. And finally, in Section 7, we define an order-isomorphism from the set of all Tychonoff extensions of a Tychonoff space  $X$  into the set of all ideals of  $C^*(X)$ , partially ordered with inclusion. Using this, we show how some of our previous results may be applied to obtain relations between the order structure of certain subfamilies of ideals of  $C^*(X)$ , partially ordered with inclusion, and the topology of subspaces of  $X^*$ .

## 2. THE PARTIALLY ORDERED SET OF ONE-POINT TYCHONOFF EXTENSIONS OF A LOCALLY COMPACT SPACE

The purpose of this article is to study the order structure of various sets of one-point extensions. To make reference to these sets easier, we list them all in the following definition.

**Definition 2.1.** For a Tychonoff space  $X$ , let  $\mathcal{T}(X)$  denote the set of all one-point Tychonoff extensions of  $X$ . We define

- $\mathcal{T}_C(X) = \{Y \in \mathcal{T}(X) : Y \text{ is Čech-complete}\}$
- $\mathcal{T}_K(X) = \{Y \in \mathcal{T}(X) : Y \text{ is locally compact}\}$
- $\mathcal{T}_D(X) = \{Y = X \cup \{p\} \in \mathcal{T}(X) : p \notin \text{cl}_Y A \text{ for any closed Lindelöf } A \subseteq X\}$
- $\mathcal{T}_L(X) = \{Y \in \mathcal{T}(X) : Y \text{ is Lindelöf}\}$
- $\mathcal{T}_S(X) = \{Y = X \cup \{p\} \in \mathcal{T}(X) : U \setminus \{p\} \text{ is } \sigma\text{-compact for some neighborhood } U \text{ of } p \text{ in } Y\}$
- $\mathcal{T}_P(X) = \{Y \in \mathcal{T}(X) : Y \text{ is pseudocompact}\}$
- $\mathcal{T}^*(X) = \{Y = X \cup \{p\} \in \mathcal{T}(X) : Y \text{ is first countable at } p\}$
- $\mathcal{T}_C^*(X) = \mathcal{T}^*(X) \cap \mathcal{T}_C(X)$ ,  $\mathcal{T}_K^*(X) = \mathcal{T}^*(X) \cap \mathcal{T}_K(X)$ ,  $\mathcal{T}_S^*(X) = \mathcal{T}^*(X) \cap \mathcal{T}_S(X)$ ,  $\mathcal{T}_{CL}(X) = \mathcal{T}_C(X) \cap \mathcal{T}_L(X)$  and  $\mathcal{T}_{KL}(X) = \mathcal{T}_K(X) \cap \mathcal{T}_L(X)$ .

For a space  $X$ , let  $\mathcal{C}(X)$  denote the set of all closed subsets of  $X$ . Suppose that  $X$  is a locally compact space. For each  $Y = X \cup \{p\} \in \mathcal{T}(X)$  let  $F_Y : \beta X \rightarrow \beta Y$  be the unique continuous function such that  $F_Y|_X = \text{id}_X$ . Define a function

$$\mu : (\mathcal{T}(X), \leq) \rightarrow (\mathcal{C}(X^*) \setminus \{\emptyset\}, \subseteq)$$

by  $\mu(Y) = F_Y^{-1}(p)$ . In the following theorem we show that the function  $\mu$  so defined is an order-anti-isomorphism. This is in fact a special case of Theorem 4.2 of [10]. We give a direct proof in here for the sake of completeness.

**Theorem 2.2.** *The function  $\mu$  is an order-anti-isomorphism.*

*Proof.* First we show that  $\mu$  is onto. So suppose that  $C \in \mathcal{C}(X^*) \setminus \{\emptyset\}$  and let  $Z$  be the space obtained from  $\beta X$  by contracting  $C$  to a point  $p$ . Let  $q : \beta X \rightarrow Z$  be the natural quotient mapping. Consider  $Y = X \cup \{p\} \subseteq Z$ . Then since  $Z$  is Tychonoff,  $Y \in \mathcal{T}(X)$ . We verify that  $\mu(Y) = C$ . First we note that  $Z = \beta Y$ . This is because  $Z$  is a compactification of  $Y$  and every continuous function from  $Y$  to  $\mathbf{I}$  is continuously extendable over  $Z$ . For if  $h \in C(Y, \mathbf{I})$ , let  $g = hq : X \cup C \rightarrow \mathbf{I}$  and let  $G \in C(\beta X, \mathbf{I})$  be the extension of  $g$ . Let the function  $H : Z \rightarrow \mathbf{I}$  be defined by  $H|(\beta X \setminus C) = G$  and  $H(p) = h(p)$ . Then  $H$  is a continuous extension of  $h$ . Now since  $q|_X = \text{id}_X = F_Y|_X$ , we have  $q = F_Y$ , and therefore  $\mu(Y) = F_Y^{-1}(p) = q^{-1}(p) = C$ .

Now suppose that  $Y_i = X \cup \{p_i\} \in \mathcal{T}(X)$ , for  $i = 1, 2$ . Suppose that  $Y_1 \geq Y_2$ , and let  $k : Y_1 \rightarrow Y_2$  be a continuous function which leaves  $X$  pointwise fixed. Let  $K : \beta Y_1 \rightarrow \beta Y_2$  be the continuous extension of  $k$ . We denote  $F_i = F_{Y_i}$ , and we let  $L = KF_1 : \beta X \rightarrow \beta Y_2$ . Then since  $L|_X = F_2|_X$ , we have  $L = F_2$  and therefore since  $K(p_1) = k(p_1) = p_2$  we obtain

$$\mu(Y_2) = F_2^{-1}(p_2) = L^{-1}(p_2) = F_1^{-1}K^{-1}(p_2) \supseteq F_1^{-1}(p_1) = \mu(Y_1).$$

Next suppose that  $\mu(Y_2) \supseteq \mu(Y_1)$  and let  $C_i = \mu(Y_i)$ , for  $i = 1, 2$ . Let  $Z_i$  be the quotient space obtained from  $\beta X$  by identifying each fiber of  $F_i = F_{Y_i}$  to a point and let  $q_i : \beta X \rightarrow Z_i$  denote its corresponding natural quotient mapping.

Let the function  $G_i : Z_i \rightarrow \beta Y_i$  be defined by  $G_i(F_i^{-1}(y)) = y$ . Then  $G_i$  is a continuous bijection, and since  $Z_i$  is compact, it is a homeomorphism which since  $F_i(X^*) = \beta Y_i \setminus X$  (see Theorem 3.5.7 of [4]) keeps  $X$  pointwise fixed and  $G_i(C_i) = p_i$ . We identify  $Y_i$  with a subspace of  $Z_i$  under this homeomorphism. Let  $f : Y_1 \rightarrow Y_2$  be a function such that  $f|_X = \text{id}_X$  and  $f(p_1) = p_2$ . We verify that  $f$  is continuous. So suppose that  $V$  is an open neighborhood of  $p_2$  in  $Z_2$ . Let  $U = (Z_1 \setminus q_1(\beta X \setminus q_2^{-1}(V))) \cap Y_1$ , which is an open subset of  $Y_1$ . Since  $q_2(C_2) = \{p_2\} \subseteq V$  and  $C_1 \subseteq C_2$ , we have  $q_1^{-1}(p_1) = C_1 \subseteq q_2^{-1}(V)$ , and thus  $p_1 \notin q_1(\beta X \setminus q_2^{-1}(V))$ . Therefore  $U$  is an open neighborhood of  $p_1$  in  $Y_1$ . Now since  $U \cap X \subseteq V \cap X$ , we have  $f(U) \subseteq V$ , and thus  $f$  is continuous at  $p_1$ . Clearly  $f$  is continuous on  $X$  and therefore  $Y_1 \geq Y_2$ . This now completes the proof.  $\square$

Let  $X$  be a locally compact space and let  $Y \in \mathcal{T}(X)$  and  $C = \mu(Y)$ . If  $Z$  is the space which is obtained from  $\beta X$  by contracting  $C$  to a point  $p$  and  $q : \beta X \rightarrow Z$  is its natural quotient mapping, then as the proof of Theorem 2.2 shows, we have  $Y = X \cup \{p\} \subseteq Z$ ,  $Z = \beta Y$  and  $q = F_Y$ .

*Remark 2.3.* If  $X$  is a locally compact metrizable space then, using the notations introduced in the introduction, we have  $\mathcal{T}^*(X) = \mathcal{E}(X)$  and  $\mu|\mathcal{T}^*(X) = \lambda$ . The first assertion follows from Theorem 2 of [1]. We verify that  $\mu|\mathcal{T}^*(X) = \lambda$ .

Suppose that  $Y = X \cup \{p\} \in \mathcal{E}(X)$  and let  $\{U_n\}_{n < \omega}$  be an extension trace in  $X$  which generates  $Y$ . Let  $C = \mu(Y)$ . We show that  $C = \bigcap_{n < \omega} U_n^* = \lambda(Y)$ . First we verify that  $C \subseteq \bigcap_{n < \omega} U_n^*$ . Suppose to the contrary that there exists an  $x \in C$  such that  $x \notin \text{cl}_{\beta X} U_n$ , for some  $n < \omega$ . Now  $U_n \cup \{p\}$  is an open neighborhood of  $p$  in  $Y$ . Let  $V$  be an open subset of  $\beta Y$  such that  $U_n \cup \{p\} = V \cap Y$ , and let  $U$  be an open neighborhood of  $x$  in  $\beta X$  such that  $F_Y(U) \subseteq V$ . Now since  $U \setminus \text{cl}_{\beta X} U_n$  contains  $x$ , it is non-empty. Let  $t \in (U \setminus \text{cl}_{\beta X} U_n) \cap X$ . Then  $t = F_Y(t) \in V$  and thus  $t \in U_n$ . But this contradicts the choice of  $t$ . This shows that  $x \in \bigcap_{n < \omega} U_n^*$ , and therefore  $C \subseteq \bigcap_{n < \omega} U_n^*$ .

To show the reverse inclusion, let  $x \in \bigcap_{n < \omega} U_n^*$ . Suppose that  $x \notin C$ . Let  $U$  and  $V$  be open subsets of  $\beta X$  such that  $x \in U$ ,  $C \subseteq V$  and  $\text{cl}_{\beta X} U \cap \text{cl}_{\beta X} V = \emptyset$ . Now since  $(V \setminus C) \cup \{p\}$  is an open neighborhood of  $p$  in  $\beta Y$ , there exists a  $k < \omega$  such that  $U_k \cup \{p\} \subseteq (V \setminus C) \cup \{p\}$ . Therefore  $\text{cl}_X U_k \subseteq V \setminus C$  and thus  $\text{cl}_X U_k \cap \text{cl}_X (U \cap X) = \emptyset$ , as

$$\text{cl}_X U_k \cap \text{cl}_X (U \cap X) \subseteq \text{cl}_{\beta X} U_k \cap \text{cl}_{\beta X} U.$$

But since  $X$  is metrizable, this implies that

$$\text{cl}_{\beta X} U_k \cap \text{cl}_{\beta X} U = \text{cl}_{\beta X} U_k \cap \text{cl}_{\beta X} (U \cap X) = \emptyset$$

which is a contradiction, as  $x \in \text{cl}_{\beta X} U_k \cap \text{cl}_{\beta X} U$ . Therefore  $x \in C$  and thus  $\bigcap_{n < \omega} U_n^* \subseteq C$ . This together with the first part shows that equality holds in the latter, which proves our assertion.

**Theorem 2.4.** *Let  $X$  be a locally compact space. Then*

$$\mu(\mathcal{T}_C(X)) = \mathcal{Z}(X^*) \setminus \{\emptyset\}.$$

*Proof.* Suppose that  $Y \in \mathcal{T}_C(X)$ . Then since  $Y$  is Čech-complete,  $Y$  is a  $G_\delta$ -set in  $\beta Y$ . Therefore  $X \cup \mu(Y) = F_Y^{-1}(Y)$  is a  $G_\delta$ -set in  $\beta X$  and thus  $\mu(Y)$  is a closed  $G_\delta$ -set in  $X^*$ . Therefore  $\mu(Y)$  is a zero-set in  $X^*$ .

For the reverse inclusion, suppose that  $D \in \mathcal{Z}(X^*) \setminus \{\emptyset\}$ . By the previous theorem  $D = \mu(Y)$ , for some  $Y \in \mathcal{T}(X)$ . Let  $X^* \setminus D = \bigcup_{n < \omega} K_n$ , where each  $K_n$  is

compact. Then we have

$$Y = F_Y\left(\beta X \setminus \bigcup_{n < \omega} K_n\right) = \bigcap_{n < \omega} (\beta Y \setminus F_Y(K_n))$$

which is a  $G_\delta$ -set in  $\beta Y$ . Thus  $Y$  is Čech-complete.  $\square$

Since for a compact space  $X$ , the order structure of either of the sets  $\mathcal{C}(X)$  or  $\mathcal{Z}(X)$  determine the topology of  $X$ , from Theorems 2.2 and 2.3 we obtain the following result.

**Theorem 2.5.** *For locally compact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $\mathcal{T}(X)$  and  $\mathcal{T}(Y)$  are order-isomorphic;
- (2)  $\mathcal{T}_C(X)$  and  $\mathcal{T}_C(Y)$  are order-isomorphic;
- (3)  $X^*$  and  $Y^*$  are homeomorphic.

**Theorem 2.6.** *Let  $X$  be a locally compact space. Then*

$$\mu(\mathcal{T}_K(X)) = \mathcal{B}(X^*) \setminus \{\emptyset\}.$$

*Proof.* Suppose that  $Y = X \cup \{p\} \in \mathcal{T}_K(X)$ . Then since  $Y$  is locally compact,  $Y$  is open in  $\beta Y$ , and thus  $F_Y^{-1}(Y)$  is open in  $\beta X$ . But since  $F_Y(X^*) = \beta Y \setminus X$ , we have  $F_Y^{-1}(Y) = X \cup F_Y^{-1}(p)$ , and therefore  $\mu(Y) = F_Y^{-1}(Y) \setminus X$  is open in  $X^*$ . Thus  $\mu(Y) \in \mathcal{B}(X^*) \setminus \{\emptyset\}$ .

To show the reverse inclusion, let  $C$  be a non-empty clopen subset of  $X^*$ . Let  $C = \mu(Y)$ , for some  $Y = X \cup \{p\} \subseteq Z$ , where  $Z$  is obtained from  $\beta X$  by contracting  $C$  to a point  $p$ . Let  $q : \beta X \rightarrow Z$  denote its natural quotient mapping. Let  $U$  be an open subset of  $\beta X$  such that  $U \cap X^* = C$ , and let  $V$  be an open neighborhood of  $C$  in  $\beta X$  with  $\text{cl}_{\beta X} V \subseteq U$ . Consider the set  $W = (V \cap X) \cup \{p\}$ . Then since  $q^{-1}(W) = V$  is open in  $\beta X$ , the set  $W$  is an open neighborhood of  $p$  in  $Y$ , and since  $W \subseteq q(\text{cl}_{\beta X} V)$ , its closure  $\text{cl}_Y W$  is compact. This shows that  $Y$  is locally compact at  $p$ , and thus  $Y \in \mathcal{T}_K(X)$ .  $\square$

For a compact zero-dimensional space  $X$ , the order structure of  $\mathcal{B}(X)$  determines the topology of  $X$ . The following is now immediate.

**Theorem 2.7.** *For strongly zero-dimensional locally compact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $\mathcal{T}_K(X)$  and  $\mathcal{T}_K(Y)$  are order-isomorphic;
- (2)  $X^*$  and  $Y^*$  are homeomorphic.

**Theorem 2.8.** *Let  $X$  be a locally compact space and let  $Y \in \mathcal{T}(X)$ . Then*

$$Y = \bigvee \{S \in \mathcal{T}_C(X) : Y \geq S\}.$$

*Proof.* This follows from Theorem 2.4 and the fact that  $\mathcal{Z}(X^*)$  is a base for closed subsets of  $X^*$ .  $\square$

The next few result will have applications in the following sections.

**Theorem 2.9.** *Let  $X$  be a locally compact space. Then*

$$\mu(\mathcal{T}^*(X)) = \{C \in \mathcal{Z}(\beta X) : C \cap X = \emptyset\} \setminus \{\emptyset\}.$$

*Proof.* Suppose that  $Y = X \cup \{p\} \in \mathcal{T}^*(X)$ . Let  $\{U_n\}_{n < \omega}$  be a base at  $p$  in  $Y$  and let  $V_n$ 's be open subsets in  $\beta Y$  such that  $U_n = V_n \cap Y$ . Let for each  $n < \omega$ ,  $f_n \in C(\beta Y, \mathbf{I})$  be such that  $f_n(p) = 0$  and  $f_n(\beta Y \setminus V_n) \subseteq \{1\}$ , and let  $S = \bigcap_{n < \omega} Z(f_n)$ . We verify that  $S = \{p\}$ . For if  $y \in S$  and  $y \neq p$ , let  $U$  and  $V$  be disjoint open neighborhoods of  $y$  and  $p$  in  $\beta Y$ , respectively. Let  $k < \omega$  be such that  $U_k \subseteq V \cap Y$ . Then  $\text{cl}_{\beta Y} V_k = \text{cl}_{\beta Y} U_k \subseteq \text{cl}_{\beta Y} V$ , and therefore since  $y \in Z(f_k)$ , we have  $y \in \text{cl}_{\beta Y} V$ . But this is a contradiction as  $y \in U$  and  $U \cap V = \emptyset$ . Therefore since  $\{p\} \in \mathcal{Z}(\beta Y)$ , we have  $\mu(Y) = F_Y^{-1}(p) \in \mathcal{Z}(\beta X)$ .

To show the reverse inclusion, let  $\emptyset \neq T \in \mathcal{Z}(\beta X)$  be such that  $T \cap X = \emptyset$ , and let  $Z$  be the space obtained from  $\beta X$  by contracting  $T$  to a point  $p$ . Let  $Y = X \cup \{p\} \subseteq Z$ , and let  $q : \beta X \rightarrow Z$  be the natural quotient mapping. For each  $n < \omega$ , let

$$U_n = ((f^{-1}([0, 1/n]) \setminus T) \cup \{p\}) \cap Y.$$

Then  $U_n$  is an open neighborhood of  $p$  in  $Y$ . Suppose that  $U$  is an open neighborhood of  $p$  in  $Y$  and let  $U = V \cap Y$ , for some open subset  $V$  of  $\beta Y$ . Then since  $p \in V$ , we have

$$\bigcap_{n < \omega} f^{-1}([0, 1/n]) = T \subseteq q^{-1}(V).$$

Therefore there exists a  $k < \omega$  such that  $f^{-1}([0, 1/k]) \subseteq q^{-1}(V)$ , and thus  $U_k \subseteq U$ . Therefore  $\{U_n\}_{n < \omega}$  is a base at  $p$  in  $Y$  and  $T = \mu(Y) \in \mu(\mathcal{T}^*(X))$ .  $\square$

**Corollary 2.10.** *Let  $X$  be a locally compact space. Then  $\mathcal{T}^*(X) \neq \emptyset$  if and only if  $X$  is not pseudocompact.*

*Proof.* We note that  $vX$  is the intersection of all cozero-sets of  $\beta X$  which contain  $X$ . But  $X$  is pseudocompact if and only if  $vX = \beta X$ . Now Theorem 2.9 completes the proof.  $\square$

**Theorem 2.11.** *Let  $X$  be a locally compact space. Then*

$$\mu(\mathcal{T}_P(X)) = \{C \in \mathcal{C}(X^*) : C \supseteq \beta X \setminus vX\} \setminus \{\emptyset\}.$$

*Proof.* Suppose that  $Y = X \cup \{p\} \in \mathcal{T}_P(X)$ . Let  $C = \mu(Y)$ . Assume that  $(\beta X \setminus vX) \setminus C \neq \emptyset$ , and let  $x \in (\beta X \setminus vX) \setminus C$ . Since  $x \notin C$ , there exists an  $S \in \mathcal{Z}(\beta X)$  such that  $x \in S$  and  $S \cap C = \emptyset$ . Since  $x \notin vX$ , there exists a  $T \in \mathcal{Z}(\beta X)$  such that  $x \in T$  and  $T \cap X = \emptyset$ . Now since  $D = (S \cap T) \setminus C$  is a non-empty  $G_\delta$ -set of  $\beta X$ , it is also a non-empty  $G_\delta$ -set of  $\beta Y$  (which is obtained from  $\beta X$  by contracting  $C$  to the point  $p$ ) and therefore by pseudocompactness of  $Y$  we have  $D \cap Y \neq \emptyset$ . But this is a contradiction as  $D \cap X = \emptyset$  and  $p \notin D$ .

To show the reverse inclusion, suppose that  $C \in \mathcal{C}(X^*) \setminus \{\emptyset\}$  and  $C \supseteq \beta X \setminus vX$ . Let  $Y = X \cup \{p\} \in \mathcal{T}(X)$  be such that  $\mu(Y) = C$ . Suppose that  $Y$  is not pseudocompact. Then there exists a non-empty  $S \in \mathcal{Z}(\beta Y)$  (note that  $\beta Y$  is obtained from  $\beta X$  by contracting  $C$  to the point  $p$  and  $q : \beta X \rightarrow \beta Y$  is its corresponding quotient mapping) such that  $S \cap Y = \emptyset$ . Now  $q^{-1}(S) \in \mathcal{Z}(\beta X)$  and since  $p \notin S$ , we have  $q^{-1}(S) \cap C = \emptyset$ . Therefore  $q^{-1}(S) \subseteq \beta X \setminus C \subseteq vX$ . Thus since  $q^{-1}(S) \in \mathcal{Z}(vX)$  and  $q^{-1}(S) \neq \emptyset$ , we have  $q^{-1}(S) \cap X \neq \emptyset$ , which is contradiction, as  $S \cap X = \emptyset$ . This shows that  $Y$  is pseudocompact, and thus  $C \in \mu(\mathcal{T}_P(X))$ . This together with the first part of the proof gives the result.  $\square$

### 3. THE CASE WHEN $X$ IS LOCALLY COMPACT AND PARACOMPACT

In this section we study the relation between the order structure of various sets of one-point extensions of a locally compact paracompact space  $X$ , and the topology of a certain subspace of  $X^*$ . We make use of the following result in a number of occasions throughout (see Theorem 5.1.27 and 3.8.C of [4]).

**Proposition 3.1.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space. Then we have*

$$X = \bigoplus_{i \in I} X_i,$$

where each  $X_i$  is a  $\sigma$ -compact non-compact subspace.

Following the notations of [7], for a Tychonoff space  $X$ , we let

$$\sigma X = \bigcup \{ \text{cl}_{\beta X} A : A \subseteq X \text{ is } \sigma\text{-compact} \}.$$

Using the notations of Proposition 3.1, it can be shown that for a locally compact paracompact non- $\sigma$ -compact space  $X$ , we have

$$\sigma X = \bigcup \left\{ \text{cl}_{\beta X} \left( \bigcup_{i \in J} X_i \right) : J \subseteq I \text{ is countable} \right\}$$

which is clearly an open subset of  $\beta X$ , as each  $\bigcup_{i \in J} X_i$  is clopen in  $X$ .

Here are some examples showing that neither of the implications, paracompactness implies local compactness, nor its converse hold. Clearly the hedgehog with an infinite number of spines provides an example of a paracompact space which is not locally compact. Now consider the space  $\sigma X$ , when  $X$  is an uncountable discrete space. Then  $\sigma X$  is locally compact, as it is open in  $\beta X$ . However, the space  $\sigma X$  is not paracompact, as it is countably compact and non-compact.

The following follows from Theorems 2.5 and 2.8.

**Lemma 3.2.** *Let  $X$  be a locally compact space. Then*

$$\mu(\mathcal{T}_K^*(X)) = \{ Z \in \mathcal{Z}(\beta X) : Z \text{ is clopen in } X^* \setminus \{\emptyset\} \}.$$

**Lemma 3.3.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space and let  $Y = X \cup \{p\} \in \mathcal{T}(X)$ . Then the following conditions are equivalent.*

- (1)  $Y \in \mathcal{T}_K^*(X)$ ;
- (2)  $p$  has a compact neighborhood  $U$  in  $Y$  such that  $U \setminus \{p\}$  is  $\sigma$ -compact.

*Proof.* (1) implies (2). Suppose that  $Y \in \mathcal{T}_K^*(X)$  and let  $\{V_n\}_{n < \omega}$  be a base at  $p$  in  $Y$ . We may assume that for each  $n < \omega$ , we have  $V_n \supseteq \text{cl}_Y V_{n+1}$  and  $\text{cl}_Y V_n$  is compact. Then for each  $n < \omega$ , the set  $\text{cl}_Y V_n \setminus V_{n+1}$  is closed in  $\text{cl}_Y V_1$ , and therefore it is compact. We have  $\text{cl}_Y V_1 \setminus \{p\} = \bigcup_{n < \omega} (\text{cl}_Y V_n \setminus V_{n+1})$ , and thus  $\text{cl}_Y V_1$  is the desired neighborhood of  $p$ .

(2) implies (1). Suppose that  $U$  is a compact neighborhood of  $p$  such that  $U \setminus \{p\}$  is  $\sigma$ -compact. Assume the notations of Proposition 3.1. Now since  $U \setminus \{p\}$  is  $\sigma$ -compact, there exists a countable  $J \subseteq I$  such that  $U \setminus \{p\} \subseteq \bigcup_{i \in J} X_i$ . By 3.8.C of [4], for each  $i \in J$  we have  $X_i = \bigcup_{n < \omega} C_n^i$ , where for each  $n < \omega$ , the set  $C_n^i$  is open in  $X$  and we have  $\text{cl}_X C_n^i \subseteq C_{n+1}^i$  and  $\text{cl}_X C_n^i$  is compact. Let

$$\{\text{cl}_X C_n^i : i \in J \text{ and } n < \omega\} = \{D_n\}_{n < \omega}$$



and consider the family

$$\mathcal{F} = \{\text{int}_Y U \setminus (D_1 \cup \cdots \cup D_n) : n < \omega\}$$

of open neighborhoods of  $p$  in  $Y$ . If  $V$  is an open neighborhood of  $p$  in  $Y$ , then since

$$U \subseteq V \cup \bigcup_{i \in J} X_i = V \cup \bigcup \{C_n^i : i \in J \text{ and } n < \omega\}$$

by compactness of  $U$ , there exists a  $k < \omega$  such that  $U \subseteq V \cup C_{n_1}^{i_1} \cup \cdots \cup C_{n_k}^{i_k}$ , and therefore for some  $n < \omega$ ,  $U \setminus (D_1 \cup \cdots \cup D_n) \subseteq V$ . This shows that  $\mathcal{F}$  is a countable base at  $p$  in  $Y$ , and since  $Y$  is locally compact, it follows that  $Y \in \mathcal{T}_K^*(X)$ .  $\square$

**Lemma 3.4.** *For any locally compact paracompact space  $X$ , we have  $\mathcal{T}_K(X) = \mathcal{T}_K^*(X)$  if and only if  $X$  is  $\sigma$ -compact.*

*Proof.* First suppose that  $X$  is  $\sigma$ -compact and let  $Y = X \cup \{p\} \in \mathcal{T}_K(X)$ . Let  $U$  be an open neighborhood of  $p$  in  $Y$  such that  $\text{cl}_Y U$  is compact. Let  $X = \bigcup_{n < \omega} C_n$ , where for each  $n < \omega$ , the set  $C_n$  is open in  $X$  and we have  $\text{cl}_X C_n \subseteq C_{n+1}$  and  $\text{cl}_X C_n$  is compact. We show that the family  $\{U \setminus \text{cl}_X C_n : n < \omega\}$  forms a base at  $p$  in  $Y$ . To show this, suppose that  $V$  is an open neighborhood of  $p$  in  $Y$ . Then since  $\{V\} \cup \{C_n : n < \omega\}$  is an open cover of the compact set  $\text{cl}_Y U$ , there exists a  $k < \omega$  such that  $\text{cl}_Y U \subseteq V \cup C_k$ . Clearly  $U \setminus \text{cl}_X C_k \subseteq V$ . This shows that  $Y$  is first-countable at  $p$ , and thus  $Y \in \mathcal{T}_K^*(X)$ .

Now suppose that  $X$  is not  $\sigma$ -compact and assume the notations of Proposition 3.1. Let  $\omega X = X \cup \{\Omega\}$  be the one-point compactification of  $X$ . Clearly  $\omega X \in \mathcal{T}_K(X)$ . But since every neighborhood  $W$  of  $\Omega$  contains all but a finite number of  $X_i$ 's, the set  $W \setminus \{\Omega\}$  is not  $\sigma$ -compact, and thus by Lemma 3.3 we have  $\omega X \notin \mathcal{T}_K^*(X)$ .  $\square$

The next three lemmas are taken from [8]. We include them in here for the sake of completeness.

**Lemma 3.5.** *Let  $X$  be a locally compact paracompact space. If  $\emptyset \neq Z \in \mathcal{Z}(\beta X)$  then  $Z \cap \sigma X \neq \emptyset$ .*

*Proof.* Suppose that  $\{x_n\}_{n < \omega}$  is an infinite sequence in  $\sigma X$ . Using the notations of Proposition 3.1, there exists a countable  $J \subseteq I$  such that  $\{x_n\}_{n < \omega} \subseteq \text{cl}_{\beta X}(\bigcup_{i \in J} X_i)$ , and therefore  $\{x_n\}_{n < \omega}$  has a limit point in  $\sigma X$ . Thus  $\sigma X$  is countably compact, and therefore pseudocompact, and  $v(\sigma X) = \beta(\sigma X) = \beta X$ . The result now follows as for any Tychonoff space  $T$ , any non-empty zero-set of  $vT$  intersects  $T$  (see Lemma 5.11(f) of [12]).  $\square$

**Lemma 3.6.** *Let  $X$  be a locally compact paracompact space. If  $\emptyset \neq Z \in \mathcal{Z}(X^*)$  then  $Z \cap \sigma X \neq \emptyset$ .*

*Proof.* Let  $S \in \mathcal{Z}(\beta X)$  be such that  $Z = S \setminus X$ . By the above lemma  $S \cap \sigma X \neq \emptyset$ . Suppose that  $S \cap (\sigma X \setminus X) = \emptyset$ . Then  $S \cap \sigma X = S \cap X$ . Let  $L = \{i \in I : S \cap X_i \neq \emptyset\}$ , where  $X_i$ 's are as in Proposition 3.1. Clearly  $L$  is finite. Observe that  $\text{cl}_{\beta X}(\bigcup_{i \in L} X_i)$  is clopen in  $\beta X$ , as  $\bigcup_{i \in L} X_i$  is clopen in  $X$ . Let  $f$  be its characteristic function which is in  $C^*(X)$ . Now since  $Z(f) \cap S \in \mathcal{Z}(\beta X)$  misses  $\sigma X$ , by the above lemma,  $Z(f) \cap S = \emptyset$ . But since  $\beta X \setminus \sigma X \subseteq Z(f)$ , we have  $Z = S \cap (\beta X \setminus \sigma X) \subseteq S \cap Z(f) = \emptyset$ , which is a contradiction. Therefore  $Z \cap (\sigma X \setminus X) = S \cap (\sigma X \setminus X) \neq \emptyset$ .  $\square$

**Lemma 3.7.** *Let  $X$  be a locally compact paracompact space and let  $S, T \in \mathcal{Z}(X^*)$ . If  $S \cap \sigma X \subseteq T \cap \sigma X$  then  $S \subseteq T$ .*

*Proof.* Suppose that  $S \setminus T \neq \emptyset$ . Let  $x \in S \setminus T$ . Let  $f \in C(\beta X, \mathbf{I})$  be such that  $f(x) = 0$  and  $f(T) \subseteq \{1\}$ . Then  $Z(f) \cap S \in \mathcal{Z}(X^*)$  is non-empty, and therefore by the above lemma,  $Z(f) \cap S \cap \sigma X \neq \emptyset$ . But this is impossible as  $Z(f) \cap S \cap \sigma X \subseteq Z(f) \cap T = \emptyset$ .  $\square$

**Lemma 3.8.** *Let  $X$  be a locally compact paracompact space. If  $Y \in \mathcal{T}_K^*(X)$  then  $\mu(Y) \subseteq \sigma X$ .*

*Proof.* Let  $C = \mu(Y)$ , for some  $Y = X \cup \{p\} \in \mathcal{T}_K^*(X)$ . By Lemma 3.3, there exists a compact neighborhood  $W$  of  $p$  in  $Y$  such that  $W \setminus \{p\}$  is  $\sigma$ -compact. We claim that  $F_Y^{-1}(p) \subseteq \text{cl}_{\beta X}(W \setminus \{p\})$ . So suppose to the contrary that there exists an  $x \in F_Y^{-1}(p)$  such that  $x \notin \text{cl}_{\beta X}(W \setminus \{p\})$ . Let  $U$  be an open neighborhood of  $x$  in  $\beta X$  which misses  $W \setminus \{p\}$ . Since  $Y$  is locally compact,  $W$  is also a neighborhood of  $p$  in  $\beta Y$ , and therefore, there exists an open neighborhood  $V$  of  $x$  in  $\beta X$  such that  $F_Y(V) \subseteq W$ . If  $t \in U \cap V \cap X$ , then  $t = F_Y(t) \in U \cap W$ , which is a contradiction. Therefore  $C = F_Y^{-1}(p) \subseteq \text{cl}_{\beta X}(W \setminus \{p\}) \subseteq \sigma X$ .  $\square$

The proof of the following is a modification of the ones we have given for Theorems 4.1 and 4.2 of [8]. Note that a space  $X$  is locally compact and  $\sigma$ -compact if and only if  $X^* \in \mathcal{Z}(\beta X)$  (see 1B of [13]). We use this fact in several different places.

**Theorem 3.9.** *For zero-dimensional locally compact paracompact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $\mathcal{T}_K^*(X)$  and  $\mathcal{T}_K^*(Y)$  are order-isomorphic;
- (2)  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

*Proof.* (1) implies (2). Suppose that condition (1) holds. Assume that one of  $X$  and  $Y$ , say  $X$ , is  $\sigma$ -compact. Suppose that  $Y$  is not  $\sigma$ -compact and let  $Y = \bigoplus_{i \in J} Y_i$ , with  $Y_i$ 's being  $\sigma$ -compact non-compact subspaces. Since by Lemma 3.4 we have  $\mathcal{T}_K^*(X) = \mathcal{T}_K(X)$ , and  $\mathcal{T}_K(X)$  has a minimum, namely its one-point compactification,  $\mathcal{T}_K^*(X)$  and thus  $\mathcal{T}_K^*(Y)$  has a minimum. Let  $T$  be the minimum of  $\mathcal{T}_K^*(Y)$ . Then since for each countable  $L \subseteq J$ , we have  $(\bigcup_{i \in L} Y_i)^* \in \mu_Y(\mathcal{T}_K^*(Y))$ , it follows that  $(\bigcup_{i \in L} Y_i)^* \subseteq \mu_Y(T)$ , and thus  $\sigma Y \setminus Y \subseteq \mu_Y(T)$ . Now by Lemma 3.7, with  $Y^*$  and  $\mu_Y(T)$  being the zero-sets, we have  $Y^* \subseteq \mu_Y(T)$ . But by Lemma 3.2, we have  $\mu_Y(T) \in \mathcal{Z}(\beta Y)$ , and therefore  $Y^* \in \mathcal{Z}(\beta Y)$ , which is a contradiction, as we assumed that  $Y$  is not  $\sigma$ -compact (see 1B of [13]). Thus  $X$  and  $Y$  are both  $\sigma$ -compact, and so by Lemma 3.4 and condition (1),  $\mathcal{T}_K(X)$  and  $\mathcal{T}_K(Y)$  are order-isomorphic. Thus since  $X$  and  $Y$  are zero-dimensional locally compact paracompact, each is strongly zero-dimensional (see Theorem 6.2.10 of [4]). Now Theorem 2.7 implies that  $\sigma X \setminus X = X^*$  and  $\sigma Y \setminus Y = Y^*$  are homeomorphic.

Next suppose that  $X$  and  $Y$  are both non- $\sigma$ -compact and let  $\phi : \mathcal{T}_K^*(X) \rightarrow \mathcal{T}_K^*(Y)$  be an order-isomorphism. Let  $g = \mu_Y \phi \mu_X^{-1} : \mu_X(\mathcal{T}_K^*(X)) \rightarrow \mu_Y(\mathcal{T}_K^*(Y))$  and let  $\omega \sigma X = \sigma X \cup \{\Omega\}$  and  $\omega \sigma Y = \sigma Y \cup \{\Omega'\}$  be one-point compactifications. We define a function  $G : \mathcal{B}(\omega \sigma X \setminus X) \rightarrow \mathcal{B}(\omega \sigma Y \setminus Y)$  between the two Boolean algebras of clopen sets, and verify that it is an order-isomorphism.

Set  $G(\emptyset) = \emptyset$  and  $G(\omega \sigma X \setminus X) = \omega \sigma Y \setminus Y$ . Let  $U \in \mathcal{B}(\omega \sigma X \setminus X)$ . If  $U \neq \emptyset$  and  $\Omega \notin U$ , then  $U$  is an open subset of  $\sigma X \setminus X$ , and therefore it is an open subset of  $X^*$ .

There exists a countable  $J \subseteq I$  such that  $U \subseteq (\bigcup_{i \in J} X_i)^*$ , where  $X = \bigoplus_{i \in I} X_i$ , with  $X_i$ 's being  $\sigma$ -compact non-compact subspaces, and thus  $U \in \mu_X(\mathcal{T}_K^*(X))$ . In this case we let  $G(U) = g(U)$ . If  $U \neq \omega\sigma X \setminus X$  and  $\Omega \in U$ , then  $(\omega\sigma X \setminus X) \setminus U \in \mu_X(\mathcal{T}_K^*(X))$ , and we let  $G(U) = (\omega\sigma Y \setminus Y) \setminus g((\omega\sigma X \setminus X) \setminus U)$ .

To show that  $G$  is an order-isomorphism, let  $U, V \in \mathcal{B}(\omega\sigma X \setminus X)$  with  $U \subseteq V$ . We may assume that  $U \neq \emptyset$  and  $V \neq \omega\sigma X \setminus X$ . We consider the following cases.

*Case 1)* Suppose that  $\Omega \notin V$ . Then clearly  $G(U) = g(U) \subseteq g(V) = G(V)$ .

*Case 2)* Suppose that  $\Omega \notin U$  and  $\Omega \in V$ . If  $G(U) \setminus G(V) \neq \emptyset$ , then  $T = g(U) \cap g((\omega\sigma X \setminus X) \setminus V) \neq \emptyset$ , and therefore by Lemma 3.2, we have  $T \in \mu_Y(\mathcal{T}_K^*(Y))$ . Let  $S \in \mu_X(\mathcal{T}_K^*(X))$  be such that  $g(S) = T$ . Then since  $g$  is an order-isomorphism,  $S \subseteq U \cap ((\omega\sigma X \setminus X) \setminus V) = \emptyset$ , which is a contradiction. Therefore  $G(U) \subseteq G(V)$ .

*Case 3)* Suppose that  $\Omega \in U$ . Then since  $(\omega\sigma X \setminus X) \setminus V \subseteq (\omega\sigma X \setminus X) \setminus U$  we have

$$G(U) = (\omega\sigma Y \setminus Y) \setminus g((\omega\sigma X \setminus X) \setminus U) \subseteq (\omega\sigma Y \setminus Y) \setminus g((\omega\sigma X \setminus X) \setminus V) = G(V).$$

This shows that  $G$  is an order-homomorphism.

To complete the proof we note that since  $\phi^{-1} : \mathcal{T}_K^*(Y) \rightarrow \mathcal{T}_K^*(X)$  also is an order-isomorphism, if we denote  $h = \mu_X \phi^{-1} \mu_Y^{-1}$ , then arguing as above,  $h$  induces an order-homomorphism  $H : \mathcal{B}(\omega\sigma Y \setminus Y) \rightarrow \mathcal{B}(\omega\sigma X \setminus X)$ . It is then easy to see that  $H = G^{-1}$ .

Now since by Theorem 6.2.10 of [4] the spaces  $X$  and  $Y$  are strongly zero-dimensional,  $\sigma X$  and  $\sigma Y$ , and therefore their one-point compactifications  $\omega\sigma X$  and  $\omega\sigma Y$ , also are zero-dimensional. Thus by Stone Duality, there exists a homeomorphism  $f : \omega\sigma X \setminus X \rightarrow \omega\sigma Y \setminus Y$  such that  $f(U) = G(U)$ , for every  $U \in \mathcal{B}(\omega\sigma X \setminus X)$ . Now since for every countable  $J \subseteq I$ , we have  $\Omega' \notin g(Q_J) = G(Q_J) = f(Q_J)$ , where  $Q_J = (\bigcup_{i \in J} X_i)^*$ , the function  $f|(\sigma X \setminus X) : \sigma X \setminus X \rightarrow \sigma Y \setminus Y$  is a homeomorphism.

(2) *implies (1)*. Suppose that condition (2) holds. If one of  $X$  and  $Y$ , say  $X$ , is  $\sigma$ -compact, then since  $\sigma Y \setminus Y$  and  $X^* = \sigma X \setminus X$  are homeomorphic,  $\sigma Y \setminus Y$  is compact. Suppose that  $Y$  is not  $\sigma$ -compact and let  $Y_i$ 's be as in the previous part. By compactness of  $\sigma Y \setminus Y$ , there exists a countable  $L \subseteq J$  such that  $(\bigcup_{i \in L} Y_i)^* = \sigma Y \setminus Y$ , which is clearly false. Thus  $Y$  also is  $\sigma$ -compact, and since  $X^*$  and  $Y^*$  are homeomorphic, by Theorem 2.7 and Lemma 3.4 we have that  $\mathcal{T}_K^*(X)$  and  $\mathcal{T}_K^*(Y)$  are order-isomorphic.

Next suppose that  $X$  and  $Y$  are both non- $\sigma$ -compact and let  $f : \sigma X \setminus X \rightarrow \sigma Y \setminus Y$  be a homeomorphism. Let  $Z \in \mu_X(\mathcal{T}_K^*(X))$ . Then by Lemma 3.8, we have  $Z \subseteq \sigma X \setminus X$ , and thus there exists a countable  $A \subseteq I$  such that  $Z \subseteq \text{cl}_{\beta X} P$ , where  $P = \bigcup_{i \in A} X_i$ . But since  $P^*$  is clopen in  $\sigma X \setminus X$ ,  $f(P^*)$  is clopen in  $\sigma Y \setminus Y$ , and since it is also compact, there exists a countable  $B \subseteq J$  such that  $f(P^*) \subseteq Q^*$ , where  $Q = \bigcup_{i \in B} Y_i$  and  $Y = \bigoplus_{i \in J} Y_i$ , with each  $Y_i$  being a  $\sigma$ -compact non-compact subspace. Since by Lemma 3.2, the set  $Z$  is clopen in  $X^*$ , the set  $f(Z)$  is clopen in  $\sigma Y \setminus Y$ , and since we have  $f(Z) \subseteq Q^*$ , it also is clopen in  $Q^*$ , and thus clopen in  $Y^*$ , i.e.,  $f(Z) \in \mu_Y(\mathcal{T}_K^*(Y))$ . Now we define a function  $F : \mu_X(\mathcal{T}_K^*(X)) \rightarrow \mu_Y(\mathcal{T}_K^*(Y))$  by  $F(Z) = f(Z)$ . The function  $F$  is clearly well-defined and it is an order-homomorphism. Since  $f^{-1}$  is also a homeomorphism, arguing as above, we can define a function  $G : \mu_Y(\mathcal{T}_K^*(Y)) \rightarrow \mu_X(\mathcal{T}_K^*(X))$  by  $G(Z) = f^{-1}(Z)$ , which is clearly the inverse of  $F$ . Thus  $F$  is an order-isomorphism.  $\square$

The following question naturally arises in connection with Theorem 3.9 above.

**Question 3.10.** *Is there any subset of  $X^*$  whose topology determines and is determined by the order structure of  $\mathcal{T}^*(X)$ ? (See Theorem 5.10 for a partial answer to this question)*

We note that for a locally compact space  $X$ , each Lindelöf subspace of  $X$  is a subset of a  $\sigma$ -compact subset of  $X$ , and therefore we can describe the elements of  $\mathcal{T}_D(X)$  as those  $Y = X \cup \{p\} \in \mathcal{T}(X)$  for which  $p \notin \text{cl}_Y A$ , for any  $\sigma$ -compact  $A \subseteq X$ .

**Theorem 3.11.** *Let  $X$  be a locally compact paracompact space. Then*

$$\mu(\mathcal{T}_D(X)) = \mathcal{C}(\beta X \setminus \sigma X) \setminus \{\emptyset\}.$$

*Proof.* Suppose that  $Y = X \cup \{p\} \in \mathcal{T}_D(X)$  and let  $C = F_Y^{-1}(p)$ . Assume that  $C \cap \sigma X \neq \emptyset$  and let  $x \in C \cap \sigma X$ . Let  $A$  be a  $\sigma$ -compact subset of  $X$  such that  $x \in \text{cl}_{\beta X} A$ . By assumption  $p \notin \text{cl}_Y A$ . Therefore  $U \cap Y \cap A = \emptyset$ , for some open neighborhood  $U$  of  $p$  in  $\beta Y$ . Now since  $F_Y^{-1}(U)$  is an open neighborhood of  $x$  in  $\beta X$ , we have  $A \cap F_Y^{-1}(U) \neq \emptyset$ . Let  $a \in A \cap F_Y^{-1}(U)$ . Then  $a = F_Y(a) \in U \cap A$ , which is a contradiction. Therefore  $\mu(Y) = C \subseteq \beta X \setminus \sigma X$ .

Conversely, suppose that  $C \in \mathcal{C}(\beta X \setminus \sigma X) \setminus \{\emptyset\}$ . Then since  $\sigma X$  is open in  $\beta X$ , we have  $C \in \mathcal{C}(X^*)$ , and thus  $C = \mu(Y)$ , for some  $Y = X \cup \{p\} \in \mathcal{T}(X)$ . Suppose that  $A \subseteq X$  is  $\sigma$ -compact. Then since  $\text{cl}_{\beta X} A \subseteq \sigma X$ , we have  $C \cap \text{cl}_{\beta X} A = \emptyset$ . Let  $U = (\beta X \setminus (C \cup \text{cl}_{\beta X} A)) \cup \{p\}$ . Then  $U \cap Y$  is an open neighborhood of  $p$  in  $Y$ , and  $U \cap Y \cap A = \emptyset$ . Therefore  $p \notin \text{cl}_Y A$ , which shows that  $Y \in \mathcal{T}_D(X)$ .  $\square$

**Lemma 3.12.** *Suppose that  $X$  is a locally compact paracompact space and let  $Y \in \mathcal{T}(X)$ . Then  $Y \in \mathcal{T}_L(X)$  if and only if  $\mu(Y) \supseteq \beta X \setminus \sigma X$ .*

*Proof.* Clearly it suffices to consider only the case when  $X$  is non- $\sigma$ -compact. Suppose that  $Y \in \mathcal{T}_L(X)$  and let  $C = \mu(Y)$ . Assume that  $(\beta X \setminus \sigma X) \setminus C \neq \emptyset$  and let  $x \in (\beta X \setminus \sigma X) \setminus C \neq \emptyset$ . Let  $U$  and  $V$  be disjoint open neighborhoods of  $x$  and  $C$  in  $\beta X$ , respectively. Assume the notations of Proposition 3.1 and let

$$J = \{i \in I : X_i \cap U \neq \emptyset\}.$$

Clearly

$$\text{cl}_{\beta X} U = \text{cl}_{\beta X} (U \cap X) \subseteq \text{cl}_{\beta X} \left( \bigcup_{i \in J} X_i \right)$$

and thus since  $x \notin \sigma X$ , the set  $J$  is uncountable. Then  $X \setminus V$ , being closed in the Lindelöf space  $Y$ , is Lindelöf. But this is a contradiction, as since  $U$  intersects uncountably many of  $X_i$ 's, there is no countable subcover of  $\{X_i\}_{i \in I}$  covering  $X \setminus V$ . Therefore  $C \supseteq \beta X \setminus \sigma X$ .

To prove the converse, suppose that  $\mu(Y) = C \supseteq \beta X \setminus \sigma X$ . Let  $\mathcal{V}$  be an open cover of  $Y = X \cup \{p\}$ . Let  $V \in \mathcal{V}$  be such that  $p \in V$ , and let  $W$  be an open set in  $\beta Y$  such that  $V = W \cap Y$ . Then since  $p \in W$ , we have  $\beta X \setminus F_Y^{-1}(W) \subseteq \sigma X$ , and therefore  $\beta X \setminus F_Y^{-1}(W) \subseteq \text{cl}_{\beta X} M$ , where  $M = \bigcup_{i \in J} X_i$  and  $J \subseteq I$  is countable. Clearly  $Y \setminus V \subseteq M$ . But  $M$ , being  $\sigma$ -compact, can be covered by countably many subsets of  $\mathcal{V}$ . Therefore  $\mathcal{V}$  has a countable subcover, which shows that  $Y$  is Lindelöf.  $\square$

**Theorem 3.13.** *Let  $X$  be a locally compact paracompact non-Lindelöf space. Then the minimum of  $\mathcal{T}_D(X)$  is the unique Lindelöf element of  $\mathcal{T}_D(X)$ .*

*Proof.* Let  $Y \in \mathcal{T}(X)$  be such that  $\mu(Y) = \beta X \setminus \sigma X$ . Then by Theorem 3.11 we have  $Y \in \mathcal{T}_D(X)$ , and by Lemma 3.12 the space  $Y$  is Lindelöf. Suppose that  $S \in \mathcal{T}_D(X)$  is Lindelöf. Then by the above lemma we have  $\mu(S) \supseteq \beta X \setminus \sigma X$ , and thus  $S = Y$ .  $\square$

**Theorem 3.14.** *For locally compact paracompact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $\mathcal{T}_D(X)$  and  $\mathcal{T}_D(Y)$  are order-isomorphic;
- (2)  $\beta X \setminus \sigma X$  and  $\beta Y \setminus \sigma Y$  are homeomorphic.

For a Tychonoff space  $X$  and a cardinal number  $\alpha$ , let  $\mathcal{T}_\alpha(X)$  consist of exactly those  $Y = X \cup \{p\} \in \mathcal{T}(X)$  such that  $p \notin \text{cl}_Y(\bigcup \mathcal{F})$ , for any discrete family  $\mathcal{F}$  of compact open subsets of  $X$  with  $|\mathcal{F}| = \alpha$ . We also let  $\tau_\alpha X$  denote the set

$$X \cup \bigcup \left\{ \text{cl}_{\beta X} \left( \bigcup \mathcal{F} \right) : \mathcal{F} \text{ is a discrete family of compact open subsets of } X \text{ with } |\mathcal{F}| = \alpha \right\}.$$

Clearly, when  $X$  is locally compact,  $\tau_\alpha X$  is an open subset of  $\beta X$ . If  $X$  is a zero-dimensional locally compact paracompact non- $\sigma$ -compact space, then  $\sigma X = \tau_\omega X$ . This is because, assuming the notations of Proposition 3.1, by 3.8.C of [4], for each  $i \in I$  we have  $X_i = \bigcup_{n < \omega} C_n^i$ , where each  $C_n^i$  is open in  $X$ , such that  $\text{cl}_X C_n^i \subseteq C_{n+1}^i$  and  $\text{cl}_X C_n^i$  is compact. Let  $D_n^i$  be a clopen subset of  $X$  such that  $C_n^i \subseteq D_n^i \subseteq C_{n+1}^i$ . Then since each  $X_i = D_1^i \oplus \bigoplus_{n \geq 1} (D_{n+1}^i \setminus D_n^i)$ , the space  $X$  is a sum of compact open subsets, and thus  $\tau_\omega X = \sigma X$ .

Now using the same proof as the one we applied for Theorem 3.11 we obtain the following result.

**Theorem 3.15.** *et  $X$  be a locally compact space and let  $\alpha$  be a cardinal number. Then*

$$\mu(\mathcal{T}_\alpha(X)) = \mathcal{C}(\beta X \setminus \tau_\alpha X) \setminus \{\emptyset\}.$$

**Theorem 3.16.** *For locally compact spaces  $X$  and  $Y$  and a cardinal number  $\alpha$  the following conditions are equivalent.*

- (1)  $\mathcal{T}_\alpha(X)$  and  $\mathcal{T}_\alpha(Y)$  are order-isomorphic;
- (2)  $\beta X \setminus \tau_\alpha X$  and  $\beta Y \setminus \tau_\alpha Y$  are homeomorphic.

**Lemma 3.17.** *Let  $X$  be a locally compact paracompact space. Then*

$$\mu(\mathcal{T}_S(X)) = \{C \in \mathcal{C}(X^*) : C \subseteq \sigma X\} \setminus \{\emptyset\}.$$

*Proof.* Let  $Y = X \cup \{p\} \in \mathcal{T}(X)$  and let  $C = \mu(Y)$ . Let  $Z$  be the space obtained from  $\beta X$  by contracting  $C$  to the point  $p$ , and let  $q : \beta X \rightarrow Z = \beta Y$  be its natural quotient mapping. Suppose that  $Y = X \cup \{p\} \in \mathcal{T}_S(X)$ . Let  $U$  be a neighborhood of  $p$  in  $Y$  with  $U \setminus \{p\}$  being  $\sigma$ -compact. Then using the notations of Proposition 3.1, we have  $U \setminus \{p\} \subseteq G$ , where  $G = \bigcup_{i \in J} X_i$  and  $J \subseteq I$  is countable. We verify that  $C \subseteq G^*$ . So suppose to the contrary that there exists an  $x \in C \setminus G^*$ . Let  $V$  be an open neighborhood of  $p$  in  $\beta Y$  with  $V \cap Y \subseteq U$ . Then since  $p \in V$ , we have  $C \subseteq q^{-1}(V)$ . Now since  $x \in H = q^{-1}(V) \setminus \text{cl}_{\beta X} G$ , we have  $H \cap X \neq \emptyset$ . If  $t \in H \cap X$ , then  $t = q(t) \in V \cap X \subseteq U \setminus \{p\} \subseteq G$ , which is a contradiction, as  $G \cap H = \emptyset$ . This shows that  $C \subseteq G^* \subseteq \sigma X$ .

Conversely, suppose that  $C \in \mathcal{C}(X^*) \setminus \{\emptyset\}$  is such that  $C \subseteq \sigma X$ , and let  $\mu(Y) = C$ , for some  $Y = X \cup \{p\} \in \mathcal{T}(X)$ . Let  $G = \bigcup_{i \in J} X_i$  be such that  $C \subseteq \text{cl}_{\beta X} G$ , where  $J \subseteq I$  is countable. Let  $U = ((\text{cl}_{\beta X} G \setminus C) \cup \{p\}) \cap Y$ . Then  $U$  is an open

neighborhood of  $p$  in  $Y$  such that  $U \setminus \{p\} = G$  is  $\sigma$ -compact. Therefore  $Y \in \mathcal{T}_S(X)$ .  $\square$

**Lemma 3.18.** *Let  $X$  be a locally compact paracompact space. Then*

$$\mathcal{T}_S^*(X) = \{Y = X \cup \{p\} \in \mathcal{T}^*(X) : p \text{ has a } \sigma\text{-compact neighborhood in } Y\}$$

and

$$\mu(\mathcal{T}_S^*(X)) = \{C \in \mathcal{Z}(\beta X) : C \subseteq \sigma X \setminus X\} \setminus \{\emptyset\}.$$

*Proof.* Let  $\mathcal{T}$  denote the set of all  $Y = X \cup \{p\} \in \mathcal{T}^*(X)$  such that  $p$  has a  $\sigma$ -compact neighborhood in  $Y$ . First we find  $\mu(\mathcal{T})$ . Suppose that  $Y \in \mathcal{T}$  and let  $C = \mu(Y)$ . By Theorem 2.9, we have  $C \in \mathcal{Z}(\beta X)$ . Suppose that  $\{U_n\}_{n < \omega}$  is a base at  $p$  in  $Y$ . We may assume that  $U_1 \supseteq U_2 \supseteq \dots$ . For each  $n < \omega$ , let  $U_n = V_n \cup \{p\}$ . Then since  $p$  has a  $\sigma$ -compact neighborhood in  $Y$ , there exists a  $k < \omega$  such that  $\text{cl}_Y U_k$  is  $\sigma$ -compact. We have

$$\text{cl}_X V_k = \text{cl}_Y U_k \setminus \{p\} = \bigcup_{n \geq k} ((\text{cl}_Y U_n) \setminus U_{n+1})$$

where for each  $n \geq k$ , the set  $\text{cl}_Y U_n \setminus U_{n+1}$ , being a closed subset of  $\text{cl}_Y U_k$ , is  $\sigma$ -compact. Clearly  $\text{cl}_{\beta X}(\text{cl}_X V_k) \subseteq \sigma X$ . We verify that  $C \subseteq \text{cl}_{\beta X}(\text{cl}_X V_k)$ . To this end, let  $Z = \beta Y$  be the space obtained from  $\beta X$  by contracting  $C$  to the point  $p$ , and let  $q : \beta X \rightarrow \beta Y$  be its natural quotient mapping. Suppose that  $x \in C$  is such that  $x \notin \text{cl}_{\beta X}(\text{cl}_X V_k)$ , and let  $U$  be an open neighborhood of  $x$  in  $\beta X$  such that  $U \cap \text{cl}_X V_k = \emptyset$ . Let  $V$  be an open set in  $\beta Y$  such that  $U_k = V \cap Y$ , and let  $W$  be an open neighborhood of  $x$  in  $\beta X$  with  $q(W) \subseteq V$ . Let  $y \in U \cap W \cap X$ . Then  $y = q(y) \in V$ , and thus  $y \in U_k \cap X = V_k$ . But this is a contradiction as  $y \in U$ . This shows that  $C \subseteq \sigma X$ . Thus  $\mu(\mathcal{T}) \subseteq \{C \in \mathcal{Z}(\beta X) : C \subseteq \sigma X \setminus X\} \setminus \{\emptyset\}$ .

Next suppose that  $C \in \mathcal{Z}(\beta X) \setminus \{\emptyset\}$  is such that  $C \subseteq \sigma X \setminus X$ . Let  $C = \mu(Y)$ , for some  $Y \in \mathcal{T}^*(X)$ . Since  $C \subseteq \sigma X$ , using the notations of Proposition 3.1, there exists a countable  $J \subseteq I$  such that  $C \subseteq \text{cl}_{\beta X} M$ , where  $M = \bigcup_{i \in J} X_i$ . Since  $A = (\text{cl}_{\beta X} M \setminus C) \cup \{p\}$  is open in  $\beta Y$ , as  $\beta Y$  is the quotient space of  $\beta X$  obtained by contracting  $C$  to the point  $p$ , the set  $M \cup \{p\} = A \cap Y$  is a  $\sigma$ -compact open neighborhood of  $p$  in  $Y$ . Therefore  $Y \in \mathcal{T}$ , and thus  $C \in \mu(\mathcal{T})$ .

To complete the proof we note that combining Theorem 2.9 and Lemma 3.17 we have  $\mu(\mathcal{T}_S^*(X)) = \mu(\mathcal{T}^*(X)) \cap \mu(\mathcal{T}_S(X)) = \mu(\mathcal{T})$ , from which it follows that  $\mathcal{T}_S^*(X) = \mathcal{T}$ .  $\square$

In the following we show that the order structure of  $\mathcal{T}_S^*(X)$  can determine the topology of the set  $\sigma X \setminus X$ . The proof is a slight modification of the metric case we gave in Theorem 5.10 of [8].

**Theorem 3.19.** *For locally compact paracompact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $\mathcal{T}_S^*(X)$  and  $\mathcal{T}_S^*(Y)$  are order-isomorphic;
- (2)  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

*Proof.* (1) implies (2). Suppose that only one of  $X$  and  $Y$ , say  $X$ , is  $\sigma$ -compact (non-compact). Then clearly  $\mathcal{T}_S^*(X) = \mathcal{T}^*(X)$ . Since by 1B of [13] we have  $X^* \in \mathcal{Z}(\beta X)$ , the set  $\mathcal{T}_S^*(X)$  has a minimum, namely  $\omega X$ . Now using the same line of reasoning as in Theorem 3.9 ((1) implies (2)) we get a contradiction, which shows that  $Y$  also is  $\sigma$ -compact. Therefore since  $\mathcal{T}_C(X)$  and  $\mathcal{T}_C(Y)$  are order-isomorphic, by Theorem 2.5, the spaces  $\sigma X \setminus X = X^*$  and  $\sigma Y \setminus Y = Y^*$  are homeomorphic.

Next suppose that  $X$  and  $Y$  are both non- $\sigma$ -compact. Since  $\mu_X$  and  $\mu_Y$  are both order-anti-isomorphism, by condition (1), there exists an order-isomorphism  $\phi : \mu_X(\mathcal{T}_S^*(X)) \rightarrow \mu_Y(\mathcal{T}_S^*(Y))$ . We extend  $\phi$  by letting  $\phi(\emptyset) = \emptyset$ . Let  $\omega\sigma X = \sigma X \cup \{\Omega\}$  and  $\omega\sigma Y = \sigma Y \cup \{\Omega'\}$  be one-point compactifications. We define a function

$$\psi : \mathcal{Z}(\omega\sigma X \setminus X) \rightarrow \mathcal{Z}(\omega\sigma Y \setminus Y)$$

and verify that it is an order-isomorphism.

For a  $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ , with  $\Omega \notin Z$ , assuming the notations of Proposition 3.1, since  $Z \subseteq \text{cl}_{\beta X}(\bigcup_{i \in K} X_i)$ , for some countable  $K \subseteq I$ , we have  $Z \in \mathcal{Z}(\beta X)$ , and therefore  $Z \in \mu_X(\mathcal{T}_S^*(X)) \cup \{\emptyset\}$ . In this case, let  $\psi(Z) = \phi(Z)$ .

Now suppose that  $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$  and  $\Omega \in Z$ . Then  $(\omega\sigma X \setminus X) \setminus Z$ , being a cozero-set in  $\omega\sigma X \setminus X$ , can be written as

$$(\omega\sigma X \setminus X) \setminus Z = \bigcup_{n < \omega} Z_n$$

where for each  $n < \omega$ , we have  $Z_n \in \mathcal{Z}(\omega\sigma X \setminus X)$  and  $\Omega \notin Z_n$ , and thus  $Z_n \in \mu_X(\mathcal{T}_S^*(X)) \cup \{\emptyset\}$ . We claim that  $\bigcup_{n < \omega} \phi(Z_n)$  is a cozero-set in  $\omega\sigma Y \setminus Y$ .

To show this, let  $Y = \bigoplus_{i \in J} Y_i$ , with each  $Y_i$  being a  $\sigma$ -compact non-compact subspace. Since for each  $n < \omega$ , we have  $\phi(Z_n) \subseteq \sigma Y \setminus Y$ , there exists a countable  $L \subseteq J$  such that

$$\bigcup_{n < \omega} \phi(Z_n) \subseteq \left( \bigcup_{i \in L} Y_i \right)^* = \phi(A)$$

for some  $A \in \mu_X(\mathcal{T}_S^*(X))$ . We show that

$$\phi(A \cap Z) = \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n).$$

Since for each  $n < \omega$ , we have  $A \cap Z \cap Z_n = \emptyset$ , it follows that  $\phi(A \cap Z) \cap \phi(Z_n) = \emptyset$ , and therefore  $\phi(A \cap Z) \subseteq \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)$ .

To show the converse, let  $x \in \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)$ . Since for each  $n < \omega$ , we have  $x \notin \phi(Z_n)$ , there exists a  $B \in \mathcal{Z}(\omega\sigma Y \setminus Y)$  such that  $x \in B$ , and for each  $n < \omega$ , we have  $B \cap \phi(Z_n) = \emptyset$ . If  $x \notin \phi(A \cap Z)$ , then there exists a  $C \in \mathcal{Z}(\omega\sigma Y \setminus Y)$  such that  $x \in C$  and  $C \cap \phi(A \cap Z) = \emptyset$ . Consider  $D = \phi(A) \cap B \cap C \in \mu_Y(\mathcal{T}_S^*(Y))$ , and let  $E \in \mu_X(\mathcal{T}_S^*(X))$  be such that  $\phi(E) = D$ . Then since  $\phi(E) \cap \phi(Z_n) = \emptyset$ , for each  $n < \omega$  we have  $E \cap Z_n = \emptyset$ , and therefore  $E \subseteq Z$ . On the other hand since  $\phi(E) \subseteq \phi(A)$ , we have  $E \subseteq A$  and thus  $E \subseteq A \cap Z$ . Therefore  $\phi(E) \subseteq \phi(A \cap Z)$ , which implies that  $\phi(E) = \emptyset$ , as  $\phi(E) \subseteq C$ . This contradiction shows that  $x \in \phi(A \cap Z)$ , and therefore  $\phi(A \cap Z) = \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n)$ .

Now since  $\phi(A)$  is clopen in  $\sigma Y \setminus Y$ , by definition of  $A$ , we have

$$\begin{aligned} (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n) &= \left( \phi(A) \setminus \bigcup_{n < \omega} \phi(Z_n) \right) \cup ((\omega\sigma Y \setminus Y) \setminus \phi(A)) \\ &= \phi(A \cap Z) \cup ((\omega\sigma Y \setminus Y) \setminus \phi(A)) \in \mathcal{Z}(\omega\sigma Y \setminus Y) \end{aligned}$$

and our claim is verified. In this case we define

$$\psi(Z) = (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n).$$

Next we show that  $\psi$  is well defined. So assume that

$$Z = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} S_n$$

with  $S_n \in \mu_X(\mathcal{T}_S^*(X)) \cup \{\emptyset\}$  for all  $n < \omega$ , is another representation of  $Z$ . Suppose that  $\bigcup_{n < \omega} \phi(Z_n) \neq \bigcup_{n < \omega} \phi(S_n)$ . Without any loss of generality we may assume that  $\bigcup_{n < \omega} \phi(Z_n) \setminus \bigcup_{n < \omega} \phi(S_n) \neq \emptyset$ . Let  $x \in \bigcup_{n < \omega} \phi(Z_n) \setminus \bigcup_{n < \omega} \phi(S_n)$ . Let  $m < \omega$  be such that  $x \in \phi(Z_m)$ . Then since  $x \notin \bigcup_{n < \omega} \phi(S_n)$ , there exists an  $A \in \mathcal{Z}(\omega\sigma Y \setminus Y)$  such that  $x \in A$  and  $A \cap \bigcup_{n < \omega} \phi(S_n) = \emptyset$ . Consider  $A \cap \phi(Z_m) \in \mu_Y(\mathcal{T}_S^*(Y))$ . Let  $B \in \mu_X(\mathcal{T}_S^*(X))$  be such that  $\phi(B) = A \cap \phi(Z_m)$ . Since  $\phi(B) \subseteq A$ , we have  $B \cap S_n = \emptyset$ , for all  $n < \omega$ . But  $B \subseteq Z_m \subseteq \bigcup_{n < \omega} Z_n = \bigcup_{n < \omega} S_n$ , which implies that  $B = \emptyset$ , which is a contradiction. Therefore  $\bigcup_{n < \omega} \phi(Z_n) = \bigcup_{n < \omega} \phi(S_n)$ , and  $\psi$  is well defined.

To prove that  $\psi$  is an order-isomorphism, let  $S, Z \in \mathcal{Z}(\omega\sigma X \setminus X)$  with  $S \subseteq Z$ . Assume that  $S \neq \emptyset$ . We consider the following cases.

*Case 1)* Suppose that  $\Omega \notin Z$ . Then  $\psi(S) = \phi(S) \subseteq \phi(Z) = \psi(Z)$ .

*Case 2)* Suppose that  $\Omega \notin S$  and  $\Omega \in Z$ . Let  $Z = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} Z_n$ , with  $Z_n \in \mu_X(\mathcal{T}_S^*(X)) \cup \{\emptyset\}$ , for all  $n < \omega$ . Then since  $S \subseteq Z$ , for each  $n < \omega$ ,  $S \cap Z_n = \emptyset$ , and therefore  $\phi(S) \cap \phi(Z_n) = \emptyset$ . We have

$$\psi(S) = \phi(S) \subseteq (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n) = \psi(Z).$$

*Case 3)* Suppose that  $\Omega \in S$ . Let

$$Z = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} Z_n \text{ and } S = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} S_n$$

where for each  $n < \omega$ , the sets  $S_n, Z_n \in \mu_X(\mathcal{T}_S^*(X)) \cup \{\emptyset\}$ . Since  $S \subseteq Z$  we have  $\bigcup_{n < \omega} Z_n \subseteq \bigcup_{n < \omega} S_n$ , and so

$$S = (\omega\sigma X \setminus X) \setminus \bigcup_{n < \omega} (S_n \cup Z_n).$$

Therefore

$$\psi(S) = (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} (\phi(S_n) \cup \phi(Z_n)) \subseteq (\omega\sigma Y \setminus Y) \setminus \bigcup_{n < \omega} \phi(Z_n) = \psi(Z)$$

and thus  $\psi$  is an order-homomorphism.

To show that  $\psi$  is an order-isomorphism, we note that  $\phi^{-1} : \mu_Y(\mathcal{T}_S^*(Y)) \rightarrow \mu_X(\mathcal{T}_S^*(X))$  is an order-isomorphism. Let

$$\gamma : \mathcal{Z}(\omega\sigma Y \setminus Y) \rightarrow \mathcal{Z}(\omega\sigma X \setminus X)$$

be its induced order-homomorphism defined as above. Then it is straightforward to see that  $\gamma = \psi^{-1}$ , i.e.,  $\psi$  is an order-isomorphism, and thus  $\mathcal{Z}(\omega\sigma X \setminus X)$  and  $\mathcal{Z}(\omega\sigma Y \setminus Y)$  are order-isomorphic. This implies that there exists a homeomorphism  $f : \omega\sigma X \setminus X \rightarrow \omega\sigma Y \setminus Y$  such that  $f(Z) = \psi(Z)$ , for every  $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ . Therefore for any  $M = \bigcup_{i \in L} X_i$  with  $L \subseteq I$  countable, since  $M^* \in \mathcal{Z}(\omega\sigma X \setminus X)$ , we have  $f(M^*) = \psi(M^*) = \phi(M^*) \subseteq \sigma Y \setminus Y$ . Therefore  $f(\sigma X \setminus X) \subseteq \sigma Y \setminus Y$  and thus  $f(\Omega) = \Omega'$ . This shows that  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

(2) *implies (1).* If one of  $X$  and  $Y$ , say  $X$ , is  $\sigma$ -compact, then since  $\sigma Y \setminus Y$  is homeomorphic to  $X^*$ , it is compact, and therefore as in Theorem 3.9 ((2) implies (1)) it follows that  $Y$  also is  $\sigma$ -compact. Since by 1B of [13]  $X^* \in \mathcal{Z}(\beta X)$ , we have  $\mathcal{Z}(\beta X) \subseteq \mathcal{Z}(X^*)$ , and thus from Theorem 2.9 and Lemma 3.18, we have  $\mathcal{T}_S^*(X) = \mathcal{T}_C(X)$ . Similarly  $\mathcal{T}_S^*(Y) = \mathcal{T}_C(Y)$ . Thus in this case, the result follows from Theorem 2.5.



The case when  $X$  and  $Y$  are both non- $\sigma$ -compact, follows by a slight modification of the proof we gave in Theorem 3.9 ((2) implies (1)).  $\square$

Our next result shows that the topology of  $\sigma X \setminus X$  also can be determined by the order structure of  $\mathcal{T}_{CL}(X)$ . The following lemma follows from Theorem 2.3 and Lemma 3.12.

**Lemma 3.20.** *For a locally compact paracompact space  $X$*

$$\mu(\mathcal{T}_{CL}(X)) = \{C \in \mathcal{Z}(X^*) : C \supseteq \beta X \setminus \sigma X\} \setminus \{\emptyset\}.$$

**Theorem 3.21.** *For locally compact paracompact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $\mathcal{T}_{CL}(X)$  and  $\mathcal{T}_{CL}(Y)$  are order-isomorphic;
- (2)  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

*Proof.* (1) implies (2). First suppose that one of  $X$  and  $Y$ , say  $X$ , is  $\sigma$ -compact. Then since  $\mathcal{T}(X) = \mathcal{T}_L(X)$ , we have  $\mathcal{T}_{CL}(X) = \mathcal{T}_C(X)$ . Suppose that  $Y$  is not  $\sigma$ -compact. As  $|X^*| > 1$ , there exists two disjoint non-empty zero-sets of  $X^*$ , which by Theorem 2.4 correspond to two elements of  $\mathcal{T}_C(X)$  with no common upper bound in  $\mathcal{T}_C(X)$ . But this is not true, as we are assuming that  $\mathcal{T}_C(X)$  and  $\mathcal{T}_{CL}(Y)$  are order-isomorphic, and by Lemma 3.20, any two elements of  $\mathcal{T}_{CL}(Y)$  have a common upper bound in  $\mathcal{T}_{CL}(Y)$ . The case  $|X^*| \leq 1$  is not possible, as  $X$  is not pseudocompact, as it is paracompact and non-compact (see Theorem 5.1.20 of [4]). Therefore  $Y$  is also  $\sigma$ -compact and  $\mathcal{T}_{CL}(Y) = \mathcal{T}_C(Y)$ , and thus by Theorem 2.5, the spaces  $\sigma X \setminus X = X^*$  and  $\sigma Y \setminus Y = Y^*$  are homeomorphic.

Next suppose that  $X$  and  $Y$  are both non- $\sigma$ -compact. Then by condition (1) and the fact that  $\mu_X$  and  $\mu_Y$  are order-anti-isomorphism, there exists an order-isomorphism  $\phi : \mu_X(\mathcal{T}_{CL}(X)) \rightarrow \mu_Y(\mathcal{T}_{CL}(Y))$ . Let  $\omega\sigma X = \sigma X \cup \{\Omega\}$  and  $\omega\sigma Y = \sigma Y \cup \{\Omega'\}$  be one-point compactifications. We define a function

$$\psi : \mathcal{Z}(\omega\sigma X \setminus X) \rightarrow \mathcal{Z}(\omega\sigma Y \setminus Y)$$

and verify that it is an order-isomorphism.

Let  $X = \bigoplus_{i \in I} X_i$  and  $Y = \bigoplus_{i \in J} Y_i$ , with each  $X_i$  and  $Y_i$  being a  $\sigma$ -compact non-compact subspace. Let  $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ . Suppose that  $\Omega \in Z$ . Then since  $P = (\omega\sigma X \setminus X) \setminus Z$  is a cozero-set in  $\omega\sigma X \setminus X$ , it is  $\sigma$ -compact, and thus since  $P \subseteq \sigma X \setminus X$ , we have  $P \subseteq (\bigcup_{i \in K} X_i)^*$ , for some countable  $K \subseteq I$ . Now since  $(\bigcup_{i \in K} X_i)^*$  is clopen in  $X^*$ , we have  $Q = (Z \setminus \{\Omega\}) \cup (\beta X \setminus \sigma X) \in \mathcal{Z}(X^*)$ , and thus by Lemma 3.20, we have  $Q \in \mu_X(\mathcal{T}_{CL}(X))$ . In this case we let

$$\psi(Z) = (\phi((Z \setminus \{\Omega\}) \cup (\beta X \setminus \sigma X)) \setminus (\beta Y \setminus \sigma Y)) \cup \{\Omega'\}.$$

Now suppose that  $\Omega \notin Z$ . Then  $Z \subseteq \sigma X \setminus X$  and therefore  $Z \subseteq (\bigcup_{i \in L} X_i)^*$ , for some countable  $L \subseteq I$ . Thus we have  $Z = X^* \setminus \bigcup_{n < \omega} Z_n$ , where each  $Z_n \in \mathcal{Z}(X^*)$  contains  $\beta X \setminus \sigma X$ . In this case we let

$$\psi(Z) = Y^* \setminus \bigcup_{n < \omega} \phi(Z_n).$$

We check that  $\psi$  is well-defined. So suppose that  $Z = X^* \setminus \bigcup_{n < \omega} Z_n$  is a representation for  $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$  with  $\Omega \notin Z$ , such that each  $Z_n \in \mathcal{Z}(X^*)$  contains  $\beta X \setminus \sigma X$ . Since for each  $n < \omega$ , we have  $Y^* \setminus \phi(Z_n) \subseteq \sigma Y$ , there exists a countable

$L \subseteq J$  such that for each  $n < \omega$ , we have  $Y^* \setminus \phi(Z_n) \subseteq (\bigcup_{i \in L} Y_i)^*$ . Let  $A$  be such that  $\phi(A) = Y^* \setminus (\bigcup_{i \in L} Y_i)^*$ . We claim that

$$Y^* \setminus \bigcup_{n < \omega} \phi(Z_n) = \phi(A \cup Z) \setminus \phi(A).$$

So suppose that  $x \in Y^* \setminus \bigcup_{n < \omega} \phi(Z_n)$ . If  $x \notin \phi(A \cup Z) \setminus \phi(A)$ , then since  $x \notin \phi(Z_1) \supseteq \phi(A)$ , we have  $x \notin \phi(A \cup Z)$ , and therefore there exists a  $B \in \mathcal{Z}(Y^*)$  containing  $x$  such that  $B \cap \phi(A \cup Z) = \emptyset$ , and  $B \cap \phi(Z_n) = \emptyset$ , for each  $n < \omega$ . Let  $C$  be such that  $\phi(C) = B \cup \phi(A \cup Z)$ , and for each  $n < \omega$ , let  $S_n$  be such that  $\phi(S_n) = \phi(C) \cap \phi(Z_n) = \phi(A \cup Z) \cap \phi(Z_n)$ . Then since for each  $n < \omega$ , we have  $\phi(A) \subseteq \phi(Z_n)$  and  $Z \cap Z_n = \emptyset$ , we have  $(A \cup Z) \cap Z_n = A$ . Clearly, by the way we defined  $S_n$ , we have  $S_n \subseteq (A \cup Z) \cap Z_n = A$ , and therefore  $\phi(S_n) \subseteq \phi(A)$ . But since  $\phi(A) \subseteq \phi(Z_n)$ , we have  $\phi(A) \subseteq \phi(S_n)$ , and thus for each  $n < \omega$  we have  $\phi(C \cap Z_n) \subseteq \phi(C) \cap \phi(Z_n) = \phi(S_n) = \phi(A)$ . Therefore  $C \cap Z_n \subseteq A$ , and thus  $C \setminus Z = C \cap (\bigcup_{n < \omega} Z_n) \subseteq A$ . Therefore  $C \subseteq A \cup Z$ , and we have  $B \subseteq \phi(C) \subseteq \phi(A \cup Z)$ , which is a contradiction as  $B \cap \phi(A \cup Z) = \emptyset$ . This shows that  $Y^* \setminus \bigcup_{n < \omega} \phi(Z_n) \subseteq \phi(A \cup Z) \setminus \phi(A)$ .

Now suppose that  $x \in \phi(A \cup Z) \setminus \phi(A)$ . Suppose that for some  $n < \omega$ , we have  $x \in \phi(Z_n)$ . Then  $x \in \phi(Z_n) \cap \phi(A \cup Z) = \phi(D)$ , for some  $D$ . Clearly  $D \subseteq Z_n \cap (A \cup Z) \subseteq A$ , and thus  $x \in \phi(A)$ , which is contradiction. This proves our claim that  $Y^* \setminus \bigcup_{n < \omega} \phi(Z_n) = \phi(A \cup Z) \setminus \phi(A)$ .

Now suppose that

$$Z = X^* \setminus \bigcup_{n < \omega} S_n = X^* \setminus \bigcup_{n < \omega} Z_n$$

are representations for  $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ , with  $\Omega \notin Z$ , such that each  $S_n, Z_n \in \mathcal{Z}(X^*)$  contains  $\beta X \setminus \sigma X$ . Choose a countable  $L \subseteq J$  such that

$$Y^* \setminus \phi(S_n) \subseteq \left( \bigcup_{i \in L} Y_i \right)^* \text{ and } Y^* \setminus \phi(Z_n) \subseteq \left( \bigcup_{i \in L} Y_i \right)^*$$

for each  $n < \omega$ . Then by above we have

$$Y^* \setminus \bigcup_{n < \omega} \phi(S_n) = \phi(A \cup Z) \setminus \phi(A) = Y^* \setminus \bigcup_{n < \omega} \phi(Z_n)$$

where  $A$  is such that  $\phi(A) = Y^* \setminus (\bigcup_{i \in L} Y_i)^*$ .

Next we show that  $\psi$ , as defined, is an order-isomorphism. So suppose that  $S, Z \in \mathcal{Z}(\omega\sigma X \setminus X)$  with  $S \subseteq Z$ . We consider the following cases.

*Case 1)* Suppose that  $\Omega \in S$ . Then  $\Omega \in Z$ , and clearly by the way we defined  $\psi$ , we have  $\psi(S) \subseteq \psi(Z)$ .

*Case 2)* Suppose that  $\Omega \notin S$  but  $\Omega \in Z$ . Let  $E = \phi((Z \setminus \{\Omega\}) \cup (\beta X \setminus \sigma X))$  and let  $S = X^* \setminus \bigcup_{n < \omega} S_n$ , where for each  $n < \omega$ ,  $S_n \in \mathcal{Z}(X^*)$  contains  $\beta X \setminus \sigma X$ . Clearly  $Y^* \setminus E \subseteq \sigma Y$ . Let the countable  $L \subseteq J$  be such that for each  $n < \omega$ ,

$$Y^* \setminus \phi(S_n) \subseteq \left( \bigcup_{i \in L} Y_i \right)^* \text{ and } Y^* \setminus E \subseteq \left( \bigcup_{i \in L} Y_i \right)^*.$$

Then by above  $\psi(S) = \phi(A \cup S) \setminus \phi(A)$ , where  $\phi(A) = Y^* \setminus (\bigcup_{i \in L} Y_i)^*$ . Since  $Y^* \setminus (\bigcup_{i \in L} Y_i)^* \subseteq E$ , we have  $\phi(A) \subseteq E$ , and therefore  $A \subseteq (Z \setminus \{\Omega\}) \cup (\beta X \setminus \sigma X)$ . Now we have

$$\psi(S) \subseteq \phi(A \cup S) \subseteq \phi((Z \setminus \{\Omega\}) \cup (\beta X \setminus \sigma X))$$

and thus  $\psi(S) \subseteq \psi(Z)$ .

Case 3) Suppose that  $\Omega \notin Z$ , and let

$$S = X^* \setminus \bigcup_{n < \omega} S_n \text{ and } Z = X^* \setminus \bigcup_{n < \omega} Z_n$$

where for each  $n < \omega$ , each of  $S_n, Z_n \in \mathcal{Z}(X^*)$  contain  $\beta X \setminus \sigma X$ . Since now

$$S = X^* \setminus \bigcup_{n < \omega} (S_n \cup Z_n)$$

we have

$$\psi(S) = Y^* \setminus \bigcup_{n < \omega} \phi(S_n \cup Z_n) \subseteq Y^* \setminus \bigcup_{n < \omega} \phi(Z_n) = \psi(Z).$$

This shows that  $\psi$  is an order-homomorphism. We note that since

$$\phi^{-1} : \mu_Y(\mathcal{T}_{CL}(Y)) \rightarrow \mu_X(\mathcal{T}_{CL}(X))$$

also is an order-isomorphism, if we denote by  $\gamma : \mathcal{Z}(\omega\sigma Y \setminus Y) \rightarrow \mathcal{Z}(\omega\sigma X \setminus X)$  its induced order-homomorphism as defined above, then it is easy to see that  $\gamma = \psi^{-1}$  and thus  $\psi$  is an order-isomorphism. Let  $f : \omega\sigma X \setminus X \rightarrow \omega\sigma Y \setminus Y$  be a homeomorphism such that  $f(Z) = \psi(Z)$ , for any  $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ . Then since for each countable  $L \subseteq J$ , we have

$$f\left(\left(\bigcup_{i \in L} X_i\right)^*\right) = \psi\left(\left(\bigcup_{i \in L} X_i\right)^*\right) \subseteq \sigma Y \setminus Y$$

it follows that  $f(\sigma X \setminus X) = \sigma Y \setminus Y$ , and therefore  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

(2) *implies (1)*. If one of  $X$  and  $Y$ , say  $X$ , is  $\sigma$ -compact, then  $\sigma Y \setminus Y$ , being homeomorphic to  $X^* = \sigma X \setminus X$ , is compact, and thus  $Y$  is also  $\sigma$ -compact. Thus by Theorem 2.5, condition (1) holds.

Now suppose that both  $X$  and  $Y$  are non- $\sigma$ -compact, and let  $f : \sigma X \setminus X \rightarrow \sigma Y \setminus Y$  be a homeomorphism. We define an order-isomorphism  $\phi : \mu_X(\mathcal{T}_{CL}(X)) \rightarrow \mu_Y(\mathcal{T}_{CL}(Y))$ . Let  $D \in \mu_X(\mathcal{T}_{CL}(X))$ . By Lemma 3.20 we have  $D \in \mathcal{Z}(X^*)$  and  $D \supseteq \beta X \setminus \sigma X$ . Then since  $X^* \setminus D \subseteq \sigma X$  is  $\sigma$ -compact, using the notations of Proposition 3.1, there exists a countable  $L \subseteq I$  such that  $X^* \setminus D \subseteq (\bigcup_{i \in L} X_i)^* = A$ . Now since  $D \cap A \in \mathcal{Z}(A)$ , we have  $f(D \cap A) \in \mathcal{Z}(f(A))$ . But  $A$  is open in  $\sigma X \setminus X$ , and therefore  $f(A)$  is open in  $\sigma Y \setminus Y$ , and thus in  $Y^*$ , i.e.,  $f(A)$  is clopen in  $Y^*$ . Therefore  $B = f(D \cap A) \cup (Y^* \setminus f(A)) \in \mathcal{Z}(Y^*)$ . Let  $\phi(D) = f(D \cap (\sigma X \setminus X)) \cup (\beta Y \setminus Y)$ . It is straightforward to check that  $\phi(D) = G$ , and thus  $\phi$  is well-defined. The function  $\phi$  is clearly an order-homomorphism. If we let  $\psi : \mu_Y(\mathcal{T}_{CL}(Y)) \rightarrow \mu_X(\mathcal{T}_{CL}(X))$  be defined by  $\psi(D) = f^{-1}(D \cap (\sigma Y \setminus Y)) \cup (\beta X \setminus X)$ , then  $\psi = \phi^{-1}$ , and therefore  $\phi$  is an order-isomorphism.  $\square$

#### 4. THE RELATION BETWEEN VARIOUS SUBSETS OF ONE-POINT EXTENSIONS OF A LOCALLY COMPACT SPACE

The order-anti-isomorphism  $\mu$  enables us to obtain interesting relations between the order structure of various sets of Tychonoff extensions.

The following is a corollary of Theorems 2.3 and 2.8.

**Theorem 4.1.** *For any locally compact space  $X$  we have*

$$\mathcal{T}_C^*(X) = \mathcal{T}^*(X).$$

**Theorem 4.2.** *For any locally compact  $\sigma$ -compact space  $X$  we have*

$$\mathcal{T}^*(X) = \mathcal{T}_C(X).$$

*Proof.* Since  $X$  is locally compact and  $\sigma$ -compact, by 1B of [13] we have  $\mathcal{Z}(X^*) \subseteq \mathcal{Z}(\beta X)$ . Now the result follows from Theorems 2.3 and 2.8.  $\square$

**Theorem 4.3.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space. Then*

$$\mathcal{T}_K(X) \cap \mathcal{T}_S(X) = \mathcal{T}_K^*(X).$$

*Proof.* Suppose that  $Y = X \cup \{p\} \in \mathcal{T}_K(X) \cap \mathcal{T}_S(X)$ . Since  $Y \in \mathcal{T}_S(X)$ , there exists a closed neighborhood  $U$  of  $p$  in  $Y$  such that  $U \setminus \{p\}$  is  $\sigma$ -compact. Since  $Y \in \mathcal{T}_K(X)$ , there exists a compact neighborhood  $V$  of  $p$  in  $Y$ . Then  $U \cap V$  is a compact neighborhood of  $p$  in  $Y$  with  $(U \cap V) \setminus \{p\}$  being  $\sigma$ -compact, and therefore by Lemma 3.3 we have  $Y \in \mathcal{T}_K^*(X)$ . By Lemma 3.3, we have  $\mathcal{T}_K^*(X) \subseteq \mathcal{T}_S(X)$ , which completes the proof.  $\square$

From Theorems 2.3 and 3.11 and Lemma 3.6 we obtain the following result.

**Theorem 4.4.** *Let  $X$  be a locally compact paracompact space. Then*

$$\mathcal{T}_C(X) \cap \mathcal{T}_D(X) = \emptyset.$$

For a Tychonoff space  $X$ , if  $S, T \in \mathcal{Z}(X)$ , then  $\text{cl}_{\beta X}(S \cap T) = \text{cl}_{\beta X}S \cap \text{cl}_{\beta X}T$ . We use this fact below.

**Lemma 4.5.** *Let  $X$  be a locally compact paracompact space. If  $Z \in \mathcal{Z}(\beta X)$  is such that  $Z \cap X = \emptyset$  then  $\text{int}_{X^*}Z \subseteq \sigma X$ .*

*Proof.* Let  $Z \in \mathcal{Z}(\beta X)$  and  $Z \cap X = \emptyset$ . Suppose that  $\text{int}_{X^*}Z \setminus \sigma X \neq \emptyset$  and let  $x \in \text{int}_{X^*}Z \setminus \sigma X$ . First using the same method as in Lemma 6.4 of [7] we find a  $T \in \mathcal{Z}(X)$  such that  $x \in T^* \subseteq Z$ . Since  $\{S^* : S \in \mathcal{Z}(X)\}$  is a base for closed subsets of  $X^*$ , there exists an  $S \in \mathcal{Z}(X)$  such that  $x \in X^* \setminus S^* \subseteq Z$ . Now

$$S^* \cap \bigcap \{T^* : T \in \mathcal{Z}(X) \text{ and } x \in T^*\} = S^* \cap \{x\} = \emptyset$$

and therefore there exist  $T_1, \dots, T_n \in \mathcal{Z}(X)$  such that  $S^* \cap T_1^* \cap \dots \cap T_n^* = \emptyset$  and  $x \in T_i^*$ , for  $i = 1, \dots, n$ . Now if we let  $T = T_1 \cap \dots \cap T_n \in \mathcal{Z}(X)$ , then  $x \in T^* = T_1^* \cap \dots \cap T_n^* \subseteq X^* \setminus S^* \subseteq Z$ . Thus  $\text{cl}_{\beta X}T \subseteq Z \cup X$ . Let  $Z = Z(f)$ , for some  $f \in C(\beta X, \mathbf{I})$ . For each  $n = 1, 2, \dots$ , we have

$$\text{cl}_{\beta X}T \setminus f^{-1}([0, 1/n)) \subseteq X = \bigoplus_{i \in I} X_i$$

where each  $X_i$  is a  $\sigma$ -compact non-compact subspace. Therefore for each  $n = 1, 2, \dots$ , there exists a finite set  $J_n \subseteq I$  such that

$$\text{cl}_{\beta X}T \setminus f^{-1}([0, 1/n)) \subseteq \bigcup_{i \in J_n} X_i.$$

Let  $J = J_1 \cup J_2 \cup \dots$ . Then

$$T \subseteq \text{cl}_{\beta X}T \setminus Z \subseteq \bigcup_{n \geq 1} (\text{cl}_{\beta X}T \setminus f^{-1}([0, 1/n))) \subseteq \bigcup_{i \in J} X_i$$

and thus  $\text{cl}_{\beta X}T \subseteq \text{cl}_{\beta X}(\bigcup_{i \in J} X_i) \subseteq \sigma X$ . But this is a contradiction, as  $x \in \text{cl}_{\beta X}T \setminus \sigma X$ . This proves the lemma.  $\square$

**Lemma 4.6.** *Let  $X$  be a locally compact space. Then every zero-set of  $\beta X$  which misses  $X$  is regular-closed in  $X^*$ .*

*Proof.* Let  $Z \in \mathcal{Z}(\beta X)$  be such that  $Z \cap X = \emptyset$ , and let  $x \in Z$ . If  $x \notin \text{cl}_{X^*} \text{int}_{X^*} Z$ , then  $x \in S$ , for some  $S \in \mathcal{Z}(\beta X)$  with  $S \cap \text{cl}_{X^*} \text{int}_{X^*} Z = \emptyset$ . Let  $T = S \cap Z$ . By Lemma 15.17 of [3], for a locally compact space  $Y$ , any non-empty zero-set of  $\beta Y$  which is contained in  $Y^*$  has non-empty interior in  $Y^*$ . Therefore  $\text{int}_{X^*} T \neq \emptyset$ . But this is a contradiction, as  $\text{int}_{X^*} T \subseteq \text{int}_{X^*} Z$  and  $T \cap \text{int}_{X^*} Z = \emptyset$ . Therefore  $x \in \text{cl}_{X^*} \text{int}_{X^*} Z$  and  $Z$  is regular-closed in  $X^*$ .  $\square$

**Theorem 4.7.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space. Then  $\mathcal{T}_L(X)$  contains an order-anti-isomorphic copy of  $\mathcal{T}^*(X)$ .*

*Proof.* Suppose that  $Z \in \mu(\mathcal{T}^*(X))$ . Then by Theorem 2.9 we have  $Z \in \mathcal{Z}(\beta X)$  and  $Z \cap X = \emptyset$ , and thus by Lemma 4.5 we have  $\text{int}_{X^*} Z \subseteq \sigma X$ . Therefore  $X^* \setminus \text{int}_{X^*} Z \supseteq \beta X \setminus \sigma X$ . Define a function  $\phi : \mu(\mathcal{T}^*(X)) \rightarrow \mu(\mathcal{T}_L(X))$  by  $\phi(Z) = X^* \setminus \text{int}_{X^*} Z$ . By Lemma 3.12 the function  $\phi$  is well-defined. Clearly for  $S, T \in \mu(\mathcal{T}^*(X))$ , if  $S \subseteq T$ , then  $\phi(S) \supseteq \phi(T)$ . The converse also holds, as by Lemma 4.6 the sets  $S$  and  $T$  are regular closed in  $X^*$ . Therefore  $\phi$  and thus  $\psi = \mu^{-1} \phi \mu : \mathcal{T}^*(X) \rightarrow \mathcal{T}_L(X)$  are order-anti-isomorphism onto their images.  $\square$

**Theorem 4.8.** *Let  $X$  be a locally compact paracompact space. Then  $\mathcal{T}_L(X)$  contains an order-isomorphic copy of  $\mathcal{T}_C(X)$ .*

*Proof.* Let  $\phi : \mu(\mathcal{T}_C(X)) \rightarrow \mu(\mathcal{T}_L(X))$  be defined by  $\phi(Z) = Z \cup (\beta X \setminus \sigma X)$ . By Theorem 2.4 and Lemma 3.12 the function  $\phi$  is well-defined. If  $\phi(Z_1) \subseteq \phi(Z_2)$ , for  $Z_1, Z_2 \in \mu(\mathcal{T}_C(X))$ , then  $Z_1 \cap \sigma X \subseteq Z_2 \cap \sigma X$ , and thus by Lemma 3.7, we have  $Z_1 \subseteq Z_2$ . Therefore if we let  $\psi = \mu^{-1} \phi \mu : \mathcal{T}_C(X) \rightarrow \mathcal{T}_L(X)$ , then  $\psi$  is an order-isomorphism onto its image.  $\square$

From Theorems 4.1 and 4.8 we obtain the following.

**Corollary 4.9.** *Let  $X$  be a locally compact paracompact space. Then  $\mathcal{T}_L(X)$  contains an order-isomorphic copy of  $\mathcal{T}^*(X)$ .*

**Theorem 4.10.** *Let  $X$  be a locally compact paracompact space. Then  $\mathcal{T}_{KL}(X) \setminus \{\omega X\}$  and  $\mathcal{T}_K^*(X)$  are order-anti-isomorphic.*

*Proof.* Suppose that  $X$  is non- $\sigma$ -compact and assume the notations of Proposition 3.1. By Theorem 2.6 and Lemma 3.12 we have

$$\mu(\mathcal{T}_{KL}(X)) = \{C \in \mathcal{B}(X^*) : C \supseteq \beta X \setminus \sigma X\}.$$

Let  $\phi : \mu(\mathcal{T}_{KL}(X)) \setminus \{X^*\} \rightarrow \mu(\mathcal{T}_K^*(X))$  be defined by  $\phi(C) = X^* \setminus C$ . To see that  $\phi$  is well-defined, let  $C \in \mathcal{B}(X^*)$  be such that  $C \supseteq \beta X \setminus \sigma X$ . Then  $X^* \setminus C$ , being a compact subset of  $\sigma X$ , there exists a countable  $J \subseteq I$  such that  $X^* \setminus C \subseteq M^*$ , where  $M = \bigcup_{i \in J} X_i$ . Now  $X^* \setminus C \in \mathcal{Z}(M^*)$ , and therefore  $X^* \setminus C \in \mathcal{Z}(\text{cl}_{\beta X} M)$  as  $M$  is  $\sigma$ -compact, and thus  $X^* \setminus C \in \mathcal{Z}(\beta X)$ , as  $\text{cl}_{\beta X} M$  is clopen in  $\beta X$ . Therefore by Lemma 3.2 we have  $X^* \setminus C \in \mu(\mathcal{T}_K^*(X))$ . If  $C \in \mu(\mathcal{T}_K^*(X))$ , then by Lemma 3.8 we have  $C \subseteq \sigma X$ , and thus  $X^* \setminus C \supseteq \beta X \setminus \sigma X$ . Therefore  $X^* \setminus C \in \mu(\mathcal{T}_{KL}(X))$ . This shows that  $\phi$  is an order-anti-isomorphism which proves the lemma in this case.

When  $X$  is  $\sigma$ -compact  $\mathcal{T}_{KL}(X) = \mathcal{T}_K(X)$ , and by Lemma 3.4 we have  $\mathcal{T}_K^*(X) = \mathcal{T}_K(X)$ . Clearly in this case  $\phi$  is still a well-defined order-anti-isomorphism.  $\square$

**Corollary 4.11.** *For zero-dimensional locally compact paracompact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $\mathcal{T}_{KL}(X)$  and  $\mathcal{T}_{KL}(Y)$  are order-isomorphic;
- (2)  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

*Proof.* By the above lemma  $\mathcal{T}_{KL}(X)$  and  $\mathcal{T}_{KL}(Y)$  are order-isomorphic if and only if  $\mathcal{T}_K^*(X)$  and  $\mathcal{T}_K^*(Y)$  are order-isomorphic. Now Theorem 3.9 now completes the proof.  $\square$

**Lemma 4.12.** *Let  $X$  be a locally compact paracompact space. If  $Z \in \mathcal{Z}(X^*)$  contains  $\beta X \setminus \sigma X$ , then  $Z$  is regular-closed in  $X^*$ .*

*Proof.* We assume that  $X$  is non- $\sigma$ -compact. Suppose that  $Z \in \mathcal{Z}(X^*)$  is such that  $Z \supseteq \beta X \setminus \sigma X$ . Assume the notations of Proposition 3.1. Since  $X^* \setminus Z \subseteq \sigma X$ , and  $X^* \setminus Z$  (being a cozero-set in  $X^*$ ) is  $\sigma$ -compact, we have  $X^* \setminus Z \subseteq G^*$ , where  $G = \bigcup_{i \in J} X_i$  and  $J \subseteq I$  is countable. Obviously since  $X^* \setminus G^* \subseteq Z$  we have

$$(X^* \setminus G^*) \cup \text{cl}_{G^*} \text{int}_{G^*}(Z \cap G^*) \subseteq \text{cl}_{X^*} \text{int}_{X^*} Z.$$

To show the reverse inclusion suppose that  $x \in \text{cl}_{X^*} \text{int}_{X^*} Z$  and  $x \in G^*$ . Suppose that  $x \notin \text{cl}_{G^*} \text{int}_{G^*}(Z \cap G^*)$  and let  $V$  be an open neighborhood of  $x$  in  $G^*$  such that  $V \cap \text{int}_{G^*}(Z \cap G^*) = \emptyset$ . But since  $x \in \text{cl}_{X^*} \text{int}_{X^*} Z$ , we have  $\emptyset \neq V \cap \text{int}_{X^*} Z \subseteq G^* \cap Z$ , and thus  $V \cap \text{int}_{X^*} Z \subseteq V \cap \text{int}_{G^*}(Z \cap G^*)$ , which is a contradiction. Now since  $G$  is  $\sigma$ -compact, it is Lindelöf and therefore realcompact (see Theorem 3.11.12 of [4]). By Theorem 15.18 of [3], for a locally compact realcompact space  $T$ , any zero-set of  $T^*$  is regular-closed in  $T^*$ . Thus since  $G$  is also locally compact  $Z \cap G^* \in \mathcal{Z}(G^*)$  is regular-closed in  $G^*$ . Therefore we have

$$\text{cl}_{X^*} \text{int}_{X^*} Z = (X^* \setminus G^*) \cup \text{cl}_{G^*} \text{int}_{G^*}(Z \cap G^*) = (X^* \setminus G^*) \cup (Z \cap G^*) = Z$$

which completes the proof.  $\square$

**Theorem 4.13.** *Let  $X$  be a locally compact paracompact non-compact space. Then  $\mathcal{T}_S(X)$  contains an order-anti-isomorphic copy of  $\mathcal{T}_{CL}(X) \setminus \{\omega X\}$ .*

*Proof.* Suppose that  $X$  is non- $\sigma$ -compact and let  $\phi : \mu(\mathcal{T}_{CL}(X)) \setminus \{X^*\} \rightarrow \mu(\mathcal{T}_S(X))$  be defined by  $\phi(Z) = X^* \setminus \text{int}_{X^*} Z$ . To see that  $\phi$  is well-defined, we note that if  $Z \in \mu(\mathcal{T}_{CL}(X)) \setminus \{X^*\}$ , then by Lemma 3.20 we have  $Z \supseteq \beta X \setminus \sigma X$ , and thus since  $X^* \setminus Z \subseteq \sigma X$  is  $\sigma$ -compact, using the notations of Proposition 3.1, we have  $X^* \setminus Z \subseteq G^*$ , where  $G = \bigcup_{i \in J} X_i$  and  $J \subseteq I$  is countable. Now since  $X^* \setminus G^* \subseteq Z$ , we have  $X^* \setminus G^* \subseteq \text{int}_{X^*} Z$ , and thus  $\phi(Z) = X^* \setminus \text{int}_{X^*} Z \subseteq \sigma X$ . Therefore by Lemma 3.17 we have  $\phi(Z) \in \mu(\mathcal{T}_S(X))$ . Now since by Lemma 4.12 each  $Z \in \mu(\mathcal{T}_{CL}(X))$  is regular-closed in  $X^*$ , it follows that  $\phi$  and thus  $\psi = \mu^{-1} \phi \mu : \mathcal{T}_{CL}(X) \setminus \{\omega X\} \rightarrow \mathcal{T}_S(X)$  are order-anti-isomorphisms onto their images.  $\square$

We summarize some of the results of this section in the next theorem. For this purpose we make the following notational convention. For two partially ordered sets  $P$  and  $Q$  we write  $P \hookrightarrow Q$  ( $P$  (anti)  $\hookrightarrow Q$ , respectively) if  $Q$  contains an order-isomorphic (order-anti-isomorphic, respectively) copy of  $P$ . We write  $P \simeq Q$  ( $P$  (anti)  $\simeq Q$ , respectively) if  $P$  and  $Q$  are order-isomorphic (order-anti-isomorphic, respectively).

**Theorem 4.14.** *Let  $X$  be a locally compact paracompact space. Then*

- (1)  $\mathcal{T}^*(X)$  (anti)  $\hookrightarrow \mathcal{T}_L(X)$  (if  $X$  is non- $\sigma$ -compact);

- (2)  $\mathcal{T}_C(X) \hookrightarrow \mathcal{T}_L(X)$ ;
- (3)  $\mathcal{T}^*(X) \hookrightarrow \mathcal{T}_L(X)$ ;
- (4)  $\mathcal{T}_K^*(X) \text{ (anti)} \simeq \mathcal{T}_{KL}(X) \setminus \{\omega X\}$ ;
- (5)  $\mathcal{T}_{CL}(X) \setminus \{\omega X\} \text{ (anti)} \hookrightarrow \mathcal{T}_S(X)$ .

**Question 4.15.** In Theorems 4.7, 4.8 and 4.13, which one-point extensions constitute exactly the image of  $\psi$ ?

#### 5. THE EXISTENCE OF MINIMAL AND MAXIMAL ELEMENTS IN VARIOUS SETS OF ONE-POINT EXTENSIONS

We start this section with the following simple observation.

**Theorem 5.1.** *Let  $X$  be a locally compact non-compact space. Then the maximal elements of  $\mathcal{T}(X)$  are exactly those of the form  $X \cup \{p\} \subseteq \beta X$ , for  $p \in X^*$ . Moreover,  $\mathcal{T}(X)$  has a minimum, namely, its one-point compactification.*

**Theorem 5.2.** *Let  $X$  be a locally compact non-compact space. Then*

- (1)  $\mathcal{T}^*(X)$  has no maximal element.
- (2) The following conditions are equivalent.
  - (a)  $\mathcal{T}^*(X)$  has a minimal element;
  - (b)  $\mathcal{T}^*(X)$  has a minimum;
  - (c)  $vX$  is locally compact and  $\sigma$ -compact;
  - (d) (Hager; cited in [6], Theorem 2.9)  $X = \bigcup_{n < \omega} A_n$ , where for each  $n < \omega$ ,  $A_n$  is pseudocompact and  $A_n$  and  $X \setminus A_{n+1}$  are completely separated in  $X$ .

*Proof.* 1) Suppose to the contrary that  $Y$  is a maximal element of  $\mathcal{T}^*(X)$  and let  $S = \mu(Y)$ . By Theorem 2.9, we have  $S \in \mathcal{Z}(\beta X)$  and  $S \cap X = \emptyset$ . Clearly  $|S| = 1$ , for otherwise, there is a non-empty zero-set of  $\beta X$  properly contained in  $S$ , which contradicts the maximality of  $Y$ . Let  $T = \beta X \setminus S$ . By Theorem 15.15 of [3], for any  $\sigma$ -compact non-compact space  $G$ , we have  $|G^*| \geq 2^{2^{\aleph_0}}$ . Therefore since  $T$  is  $\sigma$ -compact non-compact, we have  $|\beta T \setminus T| \geq 2^{2^{\aleph_0}}$ . But this is clearly a contradiction, as  $\beta T \setminus T = \beta X \setminus (\beta X \setminus S) = S$ . Therefore  $\mathcal{T}^*(X)$  has no maximal element.

2) The equivalence of conditions (a) and (b) follows from the fact that by Theorem 2.9, for any  $Y_1, Y_2 \in \mathcal{T}^*(X)$  we have  $Y_1 \wedge Y_2 \in \mathcal{T}^*(X)$ .

To show that condition (b) implies (c), suppose that  $\mathcal{T}^*(X)$  has a minimum element  $Y$ . Let  $C = \mu(Y)$ . Then since by Theorem 2.9 every non-empty zero-set of  $\beta X$  which is disjoint from  $X$  corresponds to an element of  $\mathcal{T}^*(X)$ , it is contained in  $C$ , and therefore since  $vX$  is the intersection of all cozero-sets of  $\beta X$  which contain  $X$ , we have  $\beta X \setminus C \subseteq vX$ . Clearly  $vX \subseteq \beta X \setminus C$ , and therefore  $vX = \beta X \setminus C$  being a cozero-set in  $\beta X$  is  $\sigma$ -compact. It is also locally compact as it is open in  $\beta X$ . Thus condition (c) holds.

Now suppose that condition (c) holds. Then since  $vX$  is locally compact and  $\sigma$ -compact, by 1B of [13], we have  $\beta X \setminus vX \in \mathcal{Z}(\beta X)$ . We assume that  $X$  is not pseudocompact, as otherwise by Corollary 2.10 we have  $\mathcal{T}^*(X) = \emptyset$ . Let  $Y \in \mathcal{T}^*(X)$  be such that  $\mu(Y) = \beta X \setminus vX$ . Then clearly for every  $S \in \mathcal{T}^*(X)$ , we have  $vX \subseteq \beta X \setminus \mu(S)$  and thus  $\mu(S) \subseteq \mu(Y)$ , i.e.,  $Y \leq S$ , which shows that  $\mathcal{T}^*(X)$  has a minimum.  $\square$

A space is called *almost realcompact* if it is the perfect (continuous) image of a realcompact space (see [12, 6U]).

**Corollary 5.3.** *Let  $X$  be a locally compact non-compact space. Consider the following conditions.*

- (1)  $X$  is a  $P$ -space;
- (2)  $X$  is almost realcompact;
- (3)  $X$  is weakly paracompact;
- (4)  $X$  is Dieudonné-complete;
- (5)  $[MA + \neg CH]$   $X$  is perfectly normal.

*Assume that  $X$  satisfies one of the above conditions. Then  $\mathcal{T}^*(X)$  has a minimum if and only if  $X$  is  $\sigma$ -compact.*

*Proof.* Suppose that  $\mathcal{T}^*(X)$  has a minimum. First assume that one of conditions (1)-(3) and (5) holds. Then by 6AB of [12] the set  $\beta X \setminus vX$  is dense in  $X^*$ . But by Theorem 5.2  $vX$  is locally compact, and thus  $\beta X \setminus vX$  is closed in  $\beta X$ . Therefore  $\beta X \setminus vX = X^*$ , and thus  $X = vX$ , which by Theorem 5.2 is  $\sigma$ -compact.

Suppose that condition (4) holds. Then since  $\mathcal{T}^*(X)$  has a minimum, by Theorem 5.2 we have  $X = \bigcup_{n < \omega} A_n$ , where for each  $n < \omega$ ,  $A_n$  is pseudocompact. Since Dieudonné-completeness is closed hereditary, each  $\text{cl}_X A_n$  is Dieudonné-complete. But pseudocompactness and compactness coincide in the realm of Dieudonné-complete spaces (see 8.5.13 of [4]) therefore each  $\text{cl}_X A_n$  being pseudocompact is compact, and  $X = \bigcup_{n < \omega} \text{cl}_X A_n$  is  $\sigma$ -compact.

The converse is clear, as if  $X$  is  $\sigma$ -compact, then  $\omega X$  is the minimum of  $\mathcal{T}^*(X)$ .  $\square$

It is worth to note that  $X = \omega$  is the only locally compact non-compact  $P$ -space for which  $\mathcal{T}^*(X)$  has a minimum. This is because if for a locally compact non-compact  $P$ -space  $X$ ,  $\mathcal{T}^*(X)$  has a minimum, then by Theorem 5.2 we have  $X = \bigcup_{n < \omega} A_n$ , where each  $A_n$  is pseudocompact, and since each  $A_n$  is also a  $P$ -space, it is finite. Therefore  $X$  is a countable  $P$ -space, and thus it is discrete (see 4K of [5]).

**Theorem 5.4.** *Let  $X$  be a locally compact non-compact space. Then  $\mathcal{T}_C(X)$  has a minimum. If  $X$  is realcompact or paracompact then  $\mathcal{T}_C(X)$  has no maximal element.*

*Proof.* Since  $\omega X \in \mathcal{T}_C(X)$ , it is clear that  $\mathcal{T}_C(X)$  has a minimum.

Now suppose that  $X$  is realcompact. Suppose that  $\mathcal{T}_C(X)$  has a maximal element  $Y$ . If  $G = \mu(Y)$ , then  $|G| = 1$ . As otherwise,  $G$  properly contains a non-empty zero-set of  $X^*$ , contradicting the maximality of  $Y$ . Let  $G = \{p\}$  and let  $S \in \mathcal{Z}(\beta X)$  be such that  $p \in S$  and  $S \cap X = \emptyset$ . Let  $T \in \mathcal{Z}(\beta X)$  be such that  $G = T \setminus X$ . Then  $G = T \cap S \in \mathcal{Z}(\beta X)$ . Now  $\beta X \setminus G$  is almost compact and thus pseudocompact (see 6J of [5]). But it is also  $\sigma$ -compact as it is a cozero-set in  $\beta X$ , therefore, it is compact. This contradiction shows that in this case  $\mathcal{T}_C(X)$  has no maximal element.

Next suppose that  $X$  is paracompact. We may assume that  $X$  is not  $\sigma$ -compact, as  $\sigma$ -compact spaces are realcompact. Suppose that  $\mathcal{T}_C(X)$  has a maximal element  $Y$  and let  $H = \mu(Y)$ . As above  $H = \{p\}$ , for some  $p \in X^*$ . Since by Lemma 3.6  $H \cap \sigma X \neq \emptyset$ , we have  $p \in \sigma X$ . Assume the notations of Proposition 3.1 and let  $J \subseteq I$  be countable and such that  $p \in \text{cl}_{\beta X} M$ , where  $M = \bigcup_{i \in J} X_i$ . Since  $H \in \mathcal{Z}(X^*)$ , we have  $H \in \mathcal{Z}(M^*)$ . Let  $S \in \mathcal{Z}(\text{cl}_{\beta X} M)$  be such that  $H = S \cap M^*$ . Now since  $M$  is  $\sigma$ -compact,  $M^* \in \mathcal{Z}(\text{cl}_{\beta X} M)$ , and thus  $H \in \mathcal{Z}(\text{cl}_{\beta X} M)$ . But  $M^*$



is itself clopen in  $\beta X$  and therefore  $H \in \mathcal{Z}(\beta X)$ , which as in the above part we get a contradiction. Therefore  $\mathcal{T}_C(X)$  has no maximal element.  $\square$

In connection with the above theorem we remark that, assuming that every cardinal number is non-measurable, paracompact spaces are realcompact (see Corollary 5.11(m) of [12]).

**Theorem 5.5.** *Let  $X$  be a locally compact non-compact space. Then  $\mathcal{T}_K(X)$  has a minimum.  $\mathcal{T}_K(X)$  may or may not have maximal elements.*

*Proof.* It is clear that  $\mathcal{T}_K(X)$  has a minimum, namely, its one-point compactification.

Let  $X = \bigoplus_{i \in I} X_i$ , where  $I \neq \emptyset$  and for each  $i \in I$ ,  $X_i = [0, 1)$ . Since for each  $i \in I$ , we have  $X_i^* \in \mathcal{B}(X^*)$ , there exists a  $Y_i \in \mathcal{T}_K(X)$  such that  $\mu(Y_i) = X_i^*$ . Now since each  $X_i^*$  does not properly contain any non-empty element of  $\mathcal{B}(X^*)$ , the corresponding  $Y_i$ 's are maximal elements of  $\mathcal{T}_K(X)$ .

Now let  $X$  be an uncountable discrete space and let  $C \in \mathcal{B}(X^*) \setminus \{\emptyset\}$ . By Lemma 3.6, we have  $C \cap \sigma X \neq \emptyset$ . Let  $A$  be a countable subset of  $X$  such that  $C \cap A^* \neq \emptyset$ . Now  $C \cap A^*$  is clopen in  $A^* \simeq \omega^*$  and therefore it properly contains a non-empty clopen subset of  $A^*$ , which is therefore a clopen subset of  $X^*$ . By Theorem 2.6 this shows that  $\mathcal{T}_K(X)$  has no maximal element.  $\square$

**Lemma 5.6.** *Let  $X$  be a normal space. Then every one-point regular extension of  $X$  also is normal.*

*Proof.* Suppose that  $Y = X \cup \{p\}$  is a one-point regular extension of  $X$ . Let  $A$  and  $B$  be disjoint closed subsets of  $Y$ . If  $A$  and  $B$  are closed subsets of  $X$ , then obviously they can be separated by disjoint open sets in  $X$ , and thus in  $Y$ . So suppose that  $p \in A$ , and let  $U$  and  $V$  be disjoint open subsets of  $X$  such that  $A \cap X \subseteq U$  and  $B \subseteq V$ . Let  $W$  be an open neighborhood of  $p$  in  $Y$  such that  $B \cap \text{cl}_Y W = \emptyset$ . Then  $U \cup W$  and  $V \setminus \text{cl}_Y W$  are disjoint open subsets of  $Y$  which separate  $A$  and  $B$ , respectively.  $\square$

We call a space  $X$  *locally Lindelöf*, if every  $x \in X$  has an open neighborhood  $U$  in  $X$  such that  $\text{cl}_X U$  is Lindelöf.

**Theorem 5.7.** *Let  $X$  be a paracompact non-Lindelöf space. Then*

- (1)  $\mathcal{T}_D(X)$  has a minimum if and only if  $X$  is locally Lindelöf;
- (2) If  $X$  is moreover locally compact, then  $\mathcal{T}_D(X)$  has a maximal element.

*Proof.* 1) Suppose that  $X$  is locally Lindelöf. Let  $\delta X = X \cup \{\Delta\}$ , where  $\Delta \notin X$ . Define a topology on  $\delta X$  consisting of open sets of  $X$  together with sets of the form  $\{\Delta\} \cup (X \setminus F)$ , where  $F$  is a closed Lindelöf subspace of  $X$ . It is straightforward to see that  $\delta X$  is a topological space which contains  $X$  as a dense subspace. We first check that  $X$  is Hausdorff.

So suppose that  $a, b \in \delta X$  and  $a \neq b$ . If  $a, b \in X$ , then clearly they can be separated by disjoint open sets in  $X$ , and thus in  $\delta X$ . Suppose that  $a = \Delta$  and let  $U$  be an open neighborhood of  $b$  in  $X$  such that  $\text{cl}_X U$  is Lindelöf. Then the sets  $\{\Delta\} \cup (X \setminus \text{cl}_X U)$  and  $U$  are disjoint open sets of  $\delta X$  separating  $a$  and  $b$ , respectively.

Next we show that  $\delta X$  is regular. So suppose that  $y \in \delta X$  and let  $U$  be an open neighborhood of  $y$  in  $\delta X$ . First suppose that  $y = \Delta$ . Then  $U$  is of the form  $\{\Delta\} \cup (X \setminus F)$ , for some closed Lindelöf subspace  $F$  of  $X$ . For each  $x \in F$ , let

$U_x$  be an open neighborhood of  $x$  in  $X$  with  $\text{cl}_X U_x$  being Lindelöf. Since  $F$  is Lindelöf, there exist  $x_1, x_2, \dots \in F$  such that  $F \subseteq \bigcup_{n \geq 1} U_{x_n}$ . Consider the open cover  $\mathcal{U} = \{U_{x_n}\}_{n \geq 1} \cup \{X \setminus F\}$  of  $X$ . Then since  $X$  is paracompact, there exists a locally finite open refinement  $\mathcal{V}$  of  $\mathcal{U}$ . Let

$$G = \text{cl}_X \left( \bigcup \{V \in \mathcal{V} : V \cap F \neq \emptyset\} \right).$$

Then since if  $V \in \mathcal{V}$  and  $V \cap F \neq \emptyset$ , then  $V \subseteq U_{x_n}$  for some  $n \geq 1$ , and  $\mathcal{V}$  is locally finite, we have

$$G = \bigcup \{\text{cl}_X V : V \in \mathcal{V} \text{ and } V \cap F \neq \emptyset\} \subseteq \bigcup_{n \geq 1} \text{cl}_X U_{x_n} = H.$$

Thus  $G$  being a closed subset of the Lindelöf space  $H$  is itself Lindelöf. Now we note that

$$F \subseteq \bigcup \{V \in \mathcal{V} : V \cap F \neq \emptyset\} \subseteq \text{int}_X G$$

and therefore we have

$$\text{cl}_{\delta X}(\{\Delta\} \cup (X \setminus G)) = \{\Delta\} \cup \text{cl}_X(X \setminus G) \subseteq \{\Delta\} \cup (X \setminus F)$$

i.e.,  $\{\Delta\} \cup (X \setminus G)$  is an open neighborhood of  $y$  in  $\delta X$  whose closure in  $\delta X$  is contained in  $U$ . Now suppose that  $y \in X$  and let  $V$  and  $W$  be open neighborhoods of  $y$  in  $X$  with  $\text{cl}_X V$  being Lindelöf and  $\text{cl}_X W \subseteq U \cap V$ . Then  $\text{cl}_{\delta X} W = \text{cl}_X W \subseteq U$ . This shows that  $\delta X$  is regular, and since it is Lindelöf, it is normal.

Clearly  $\Delta \notin \text{cl}_{\delta X} F$ , for any closed Lindelöf subset  $F$  of  $X$ , and thus  $\delta X \in \mathcal{T}_D(X)$ . To show that  $\delta X$  is a minimum, suppose that  $Y = X \cup \{p\} \in \mathcal{T}_D(X)$  and let  $f : Y \rightarrow \delta X$  be defined such that  $f|_X = \text{id}_X$  and  $f(p) = \Delta$ . Then since any open neighborhood of  $\Delta$  in  $\delta X$  is of the form  $V = \{\Delta\} \cup (X \setminus F)$ , for some closed Lindelöf subset  $F$  of  $X$ , and  $p \notin \text{cl}_Y F$ , there exists an open neighborhood  $U$  of  $p$  in  $Y$  such that  $U \cap F = \emptyset$ , and therefore  $f(U) \subseteq V$ , i.e.,  $f$  is continuous at  $p$  and thus on  $Y$ . This shows that  $Y \geq \delta X$ , which completes the proof of this part.

Next suppose that  $\mathcal{T}_D(X)$  has a minimum, say  $Y = X \cup \{p\}$ . Suppose that  $X$  is not locally Lindelöf and let  $U$  be an open subset of  $X$  such that  $p \notin \text{cl}_Y U$  and  $\text{cl}_X U$  is not Lindelöf. Let  $\{U_i\}_{i \in I}$  be a cover of  $\text{cl}_X U$  consisting of open subsets of  $X$  with no countable subcover. Refining  $\{U_i\}_{i \in I}$  by using regularity, we may assume that  $\text{cl}_X U$  is not covered by any countable union of closures of  $U_i$ 's in  $X$ . Let  $\mathcal{V}$  be a locally finite open refinement of  $\{U_i\}_{i \in I} \cup \{X \setminus \text{cl}_X U\}$ . Let

$$\mathcal{W} = \{V \in \mathcal{V} : V \cap U \neq \emptyset\} = \{W_j\}_{j \in J}$$

which is faithfully indexed. It is clear that  $J$  is uncountable, as otherwise, since  $\{W_j\}_{j \in J}$  covers  $U$  and they are locally finite  $\text{cl}_X U \subseteq \bigcup_{i \in J} \text{cl}_X W_j$ , which is a contradiction, as each  $W_j$  is a subset of some  $U_i$ . For each  $j \in J$ , let  $x_j \in W_j \cap U$ . Let  $A = X \cup \{q\}$ , with  $q \notin X$ , and define a topology on  $A$  consisting of open sets of  $X$  together with sets of the form  $B \cup \{q\}$ , where  $B \subseteq X$ , the set  $B \cup \{p\}$  is open in  $Y$ , and  $B \supseteq \bigcup_{j \in J \setminus L} C_j$ , where  $L \subseteq J$  is countable, and for each  $j \in J \setminus L$ , the set  $C_j$  is an open neighborhood of  $x_j$  in  $X$  contained in  $W_j \cap U$ . Then it is easy to verify that  $A$  is a topological space containing  $X$  as a dense subspace.

To see that  $A$  is a  $T_1$ -space, let  $x \in X$ . Since  $\mathcal{W}$  is locally finite, there exists a finite set  $L \subseteq J$  such that  $x \notin W_j$ , for any  $j \in J \setminus L$ . Let

$$B = (D \cap X) \cup \bigcup_{j \in J \setminus L} (W_j \cap U)$$

where  $D$  is an open neighborhood of  $p$  in  $Y$  not containing  $x$ . Then  $B \cup \{q\}$  is an open neighborhood of  $q$  in  $A$  which does not contain  $x$ .

Next we show that  $A$  is regular. So suppose that  $y \in A$  and let  $W$  be an open neighborhood of  $y$  in  $A$ . First suppose that  $y = q$ . Then  $W = B \cup \{q\}$ , where  $B \subseteq X$ , the set  $B \cup \{p\}$  is open in  $Y$  and  $B \supseteq \bigcup_{j \in J \setminus L} C_j$ , for some countable  $L \subseteq J$  and open sets  $C_j$ 's of  $X$  such that  $x_j \in C_j \subseteq W_j \cap U$ . Let  $G$  be an open neighborhood of  $p$  in  $Y$  such that  $\text{cl}_Y G \subseteq B \cup \{p\}$ , and for each  $j \in J \setminus L$ , let  $H_j$  be an open neighborhood of  $x_j$  in  $X$  with  $\text{cl}_X H_j \subseteq C_j$ . Then

$$V = (G \cap X) \cup \bigcup_{j \in J \setminus L} H_j \cup \{q\}$$

is an open neighborhood of  $q$  in  $A$  and

$$\text{cl}_A V = \text{cl}_X(G \cap X) \cup \bigcup_{j \in J \setminus L} \text{cl}_X H_j \cup \{q\} \subseteq B \cup \bigcup_{j \in J \setminus L} C_j \cup \{q\} = B \cup \{q\} = W.$$

Now suppose that  $y \in X$ . Let  $F$  and  $G$  be disjoint open neighborhoods of  $p$  and  $y$  in  $Y$ , respectively. Let  $H$  be an open neighborhood of  $y$  in  $X$  such that  $\text{cl}_X H \subseteq W \cap G$ , and let  $K$  be an open neighborhood of  $y$  in  $X$  intersecting at most finitely many of  $W_j$ 's. Let the finite set  $L \subseteq J$  be such that  $K \cap W_j = \emptyset$ , for any  $j \in J \setminus L$ . Let

$$D = (F \cap X) \cup \bigcup_{j \in J \setminus L} W_j.$$

Then  $D \cup \{q\}$  is an open neighborhood of  $q$  in  $A$  missing  $K \cap H$ . Therefore  $q \notin \text{cl}_A(K \cap H)$ , and thus  $K \cap H$  is an open neighborhood of  $y$  in  $A$  such that

$$\text{cl}_A(K \cap H) = \text{cl}_X(K \cap H) \subseteq \text{cl}_X H \subseteq W.$$

This shows that  $A$  is regular and thus by Lemma 5.6, it is also normal.

Now let  $P$  be a closed Lindelöf subspace of  $X$ . For each  $x \in P \cap \text{cl}_X U$ , let  $V_x$  be an open neighborhood of  $x$  in  $X$  which intersects only finitely many of  $W_j$ 's, say for  $j \in L_x$ , where  $L_x \subseteq J$  is finite. Since  $P \cap \text{cl}_X U$  is closed in  $P$ , it is Lindelöf, and therefore since

$$P \cap \text{cl}_X U \subseteq \bigcup \{V_x : x \in P \cap \text{cl}_X U\}$$

there exist  $x_1, x_2, \dots \in P \cap \text{cl}_X U$  such that  $P \cap \text{cl}_X U \subseteq \bigcup_{n \geq 1} V_{x_n}$ . Let  $L = \bigcup_{n \geq 1} L_{x_n}$ . Then clearly for each  $j \in J \setminus L$ , we have  $W_j \cap P \cap \text{cl}_X U = \emptyset$ . Now since  $Y = X \cup \{p\} \in \mathcal{T}_D(X)$ , we have  $p \notin \text{cl}_Y P$ , and thus there exists an open neighborhood  $M$  of  $p$  in  $Y$  such that  $M \cap P = \emptyset$ . Let

$$B = (M \cap X) \cup \bigcup_{j \in J \setminus L} (W_j \cap U).$$

Then  $B \cup \{q\}$  is an open neighborhood of  $q$  in  $A$ , and we have

$$P \cap (B \cup \{q\}) = P \cap \bigcup_{j \in J \setminus L} (W_j \cap U) = \emptyset.$$

This shows that  $q \notin \text{cl}_A P$ , and thus  $A \in \mathcal{T}_D(X)$ . But this is impossible, as by the way we defined neighborhoods of  $q$  in  $A$ , each of them contains an  $x_j$ , for some  $j \in J$ , and therefore has non-empty intersection with  $U$ , which contradicts the fact that  $A \geq Y$ . This shows that  $X$  is locally Lindelöf.

2) This is clear as in this case by Theorem 3.11, any  $Y = X \cup \{p\}$ , where  $p \in \beta X \setminus \sigma X$ , belongs to  $\mathcal{T}_D(X)$  and it is obviously maximal.  $\square$

We note in passing that a hedgehog with an uncountable number of spines is an example of a paracompact space which is not locally Lindelöf.

**Theorem 5.8.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space. Then  $\mathcal{T}_L(X)$  has both maximal and minimal elements.*

*Proof.* This follows from Lemma 3.12 and the fact that both  $\beta X \setminus \sigma X$  and  $X^*$  belong to  $\mu(\mathcal{T}_L(X))$ .  $\square$

**Theorem 5.9.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space. Then  $\mathcal{T}_S(X)$  has a maximal element but does not have a minimal element.*

*Proof.* Clearly every element of the form  $Y = X \cup \{p\}$ , for  $p \in \sigma X \setminus X$ , is a maximal element of  $\mathcal{T}_S(X)$ .

Suppose that  $Y \in \mathcal{T}_S(X)$ . Assume the notations of Proposition 3.1. Then since by Lemma 3.17  $\mu(Y) \subseteq \sigma X$ , we have  $\mu(Y) \subseteq (\bigcup_{i \in J} X_i)^*$ , for some countable  $J \subseteq I$ . Let the countable  $L \subseteq I$  properly contain  $J$ . Then if  $T \in \mathcal{T}_S(X)$  is such that  $\mu(T) = (\bigcup_{i \in L} X_i)^*$ , we have  $T < Y$ . Therefore  $\mathcal{T}_S(X)$  has no minimal element.  $\square$

**Theorem 5.10.** *Let  $X$  be a locally compact non-pseudocompact space. Then  $\mathcal{T}_P(X)$  has both minimum and maximum.*

*Proof.* It is clear that  $\omega X$  is the minimum of  $\mathcal{T}_P(X)$ . Let  $C = \beta X \setminus \text{int}_{\beta X} vX$ . Then since  $X$  is locally compact  $X \subseteq \text{int}_{\beta X} vX$ , and thus  $C \subseteq X^*$ . Since  $X$  is not pseudocompact  $C \neq \emptyset$ . By Theorem 2.11 there exists a  $Y \in \mathcal{T}_P(X)$  such that  $\mu(Y) = C$ . If  $S \in \mathcal{T}_P(X)$ , then since by Theorem 2.11 we have  $\mu(S) \supseteq \beta X \setminus vX$ , it follows that  $\beta X \setminus \mu(S) \subseteq vX$ , and therefore  $\beta X \setminus \mu(S) \subseteq \text{int}_{\beta X} vX$ . Thus  $\mu(Y) \subseteq \mu(S)$ , and therefore  $S \leq Y$ . This shows that  $Y$  is maximum in  $\mathcal{T}_P(X)$ .  $\square$

**Theorem 5.11.** *Let  $X$  be a locally compact non-compact space. Then the minimum of  $\mathcal{T}^*(X)$ , if exists, is the unique pseudocompact element of  $\mathcal{T}^*(X)$  (compare with Theorem 5.5 and Corollary 5.6 of [B]).*

*Proof.* Suppose that  $Y$  is the minimum of  $\mathcal{T}^*(X)$  and let  $C = \mu(Y)$ . By the proof of Theorem 5.2 ((b) implies (c)) we know that  $C = \beta X \setminus vX$ . Therefore by Theorem 2.11, the space  $Y$  is pseudocompact.

If  $S$  is another pseudocompact element of  $\mathcal{T}^*(X)$ , then by Theorem 2.11 we have  $\mu(S) \supseteq \beta X \setminus vX$ . On the other hand, by Theorem 2.9,  $\mu(S)$  is a zero-set in  $\beta X$  contained in  $X^*$ , which implies that  $\mu(S) \subseteq \beta X \setminus vX$ . Thus  $\mu(S) = \beta X \setminus vX = \mu(Y)$ , and therefore  $S = Y$ . This shows the uniqueness of  $Y$ .  $\square$

The following result partially answers Question 3.10.

**Theorem 5.12.** *Let  $X$  and  $Y$  be locally compact non-compact spaces such that  $X = \bigcup_{n < \omega} A_n$  and  $Y = \bigcup_{n < \omega} B_n$ , where each  $A_n$  and  $B_n$  is pseudocompact and for each  $n < \omega$  the pairs  $A_n, X \setminus A_{n+1}$  and  $B_n, Y \setminus B_{n+1}$  are completely separated in  $X$ . Then the following conditions are equivalent.*

- (1)  $\mathcal{T}^*(X)$  and  $\mathcal{T}^*(Y)$  are order-isomorphic;
- (2)  $\beta X \setminus vX$  and  $\beta Y \setminus vY$  are homeomorphic.

*Proof.* By Theorem 5.2  $vX$  is locally compact and  $\sigma$ -compact, and therefore by 1B of [13], we have  $\beta X \setminus vX \in \mathcal{Z}(\beta X)$ . Let  $Z \in \mathcal{Z}(\beta X \setminus vX)$ . Then since  $vX$  is locally compact,  $\beta X \setminus vX$  is closed in  $\beta X$ , and therefore there exists an  $S \in \mathcal{Z}(\beta X)$  such that  $Z = S \cap (\beta X \setminus vX)$ . Thus  $Z \in \mathcal{Z}(\beta X)$ . Clearly  $Z \cap X = \emptyset$ , and

therefore by Theorem 2.9 we have  $\mathcal{Z}(\beta X \setminus vX) \subseteq \mu_X(\mathcal{T}^*(X)) \cup \{\emptyset\}$ . Clearly for every  $C \in \mu_X(\mathcal{T}^*(X))$ , since  $C \in \mathcal{Z}(\beta X)$  and  $C \cap X = \emptyset$ , we have  $C \subseteq \beta X \setminus vX$ . Therefore  $\mathcal{Z}(\beta X \setminus vX) = \mu_X(\mathcal{T}^*(X)) \cup \{\emptyset\}$ . Similarly  $\mathcal{Z}(\beta Y \setminus vY) = \mu_Y(\mathcal{T}^*(Y)) \cup \{\emptyset\}$ . Now since  $\mu_X$  and  $\mu_Y$  are order-anti-isomorphisms,  $\mathcal{T}^*(X)$  and  $\mathcal{T}^*(Y)$  are order-isomorphic, if and only if,  $\mathcal{Z}(\beta X \setminus vX)$  and  $\mathcal{Z}(\beta Y \setminus vY)$  are order-isomorphic, if and only if,  $\beta X \setminus vX$  and  $\beta Y \setminus vY$  are homeomorphic.  $\square$

## 6. SOME CARDINALITY THEOREMS

Suppose that  $X$  is a locally compact space. Let  $w(T)$  and  $d(T)$  denote the weight and the density of a space  $T$ , respectively. Then since

$$w(X^*) \leq w(\beta X) \leq 2^{d(\beta X)} \leq 2^{d(X)}$$

we have

$$|\mathcal{T}(X)| \leq |\mathcal{C}(X^*)| \leq 2^{w(X^*)} \leq 2^{2^{d(X)}}$$

which gives an upper bound for cardinality of the set  $\mathcal{T}(X)$ . In the following theorems we obtain a lower bound for cardinalities of two subsets of  $\mathcal{T}(X)$ . Here for a space  $T$ ,  $L(T)$  denotes the Lindelöf number of  $T$ .

**Theorem 6.1.** *Let  $X$  be a locally compact paracompact non-compact space. Then*

$$2^{L(X)} \leq |\mathcal{T}_L(X)|.$$

*Proof.* *Case 1)* Suppose that  $X$  is  $\sigma$ -compact. Then since  $X$  is non-pseudocompact, as  $X$  paracompact and non-compact (see Theorem 5.1.20 of [4]) by 4C of [13] we have  $|X^*| \geq 2^{2^{\aleph_0}}$ . Now since each element of  $\mathcal{T}(X)$  is  $\sigma$ -compact,  $\mathcal{T}_L(X) = \mathcal{T}(X)$ , and thus we have

$$|\mathcal{T}_L(X)| = |\mathcal{T}(X)| \geq |\{X \cup \{p\} : p \in X^*\}| = |X^*| \geq 2^{2^{\aleph_0}} \geq 2^{L(X)}.$$

*Case 2)* Suppose that  $X$  is non- $\sigma$ -compact. Assume the notations of Proposition 3.1. Then since each  $X_i$  is  $\sigma$ -compact, we have  $L(X) \leq |I|$ . For each  $J \subseteq I$ , let  $Q_J = \bigcup_{i \in J} X_i$  and  $C_J = Q_J^* \cup (\beta X \setminus \sigma X)$ . For  $J_1, J_2 \subseteq I$ , if  $j \in J_1 \setminus J_2$ , then since  $X_j^* \subseteq C_{J_1}$  and  $X_j^* \cap C_{J_2} = \emptyset$ , we have  $C_{J_1} \neq C_{J_2}$ . By Lemma 3.12, for each  $J \subseteq I$ , there exists  $Y_J \in \mathcal{T}_L(X)$  such that  $\mu(Y_J) = C_J$ . Now

$$|\mathcal{T}_L(X)| \geq |\mathcal{P}(I)| = 2^{|I|} \geq 2^{L(X)}.$$

$\square$

For purpose of the next result we need the following proposition stated in Lemma 35.3 of [9].

**Proposition 6.2.** *Suppose that  $E$  is an infinite set of cardinality  $\alpha$ . Then there exists a collection  $\mathcal{A}$  of subsets of  $E$  with  $|\mathcal{A}| = 2^\alpha$  such that for any distinct  $A, B \in \mathcal{A}$  we have  $|A \setminus B| = \alpha$ .*

**Theorem 6.3.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space. Then*

$$2^{L(X)} \leq |\mathcal{T}_D(X)|.$$

*Proof.* Assume the notations of Proposition 3.1. By the above proposition, since  $\alpha = |I| > \aleph_0$ , there exists a family  $\{J_s\}_{s \in S}$  of subsets of  $I$ , faithfully indexed, such that  $|S| = 2^\alpha$  and  $|J_s \setminus J_t| = \alpha$ , for distinct  $s, t \in S$ . For each  $s \in S$ , let  $Q_s = \bigcup_{i \in J_s} X_i$  and let  $C_s = Q_s^* \setminus \sigma X$ . If for some  $s \in S$ , we have  $C_s = \emptyset$ , then since  $\text{cl}_{\beta X} Q_s \subseteq \sigma X$ , we have  $\text{cl}_{\beta X} Q_s \subseteq \text{cl}_{\beta X} (\bigcup_{i \in H} X_i)$ , for some countable  $H \subseteq I$ ,

and thus  $Q_s \subseteq \bigcup_{i \in H} X_i$ , as  $\bigcup_{i \in H} X_i$  is clopen in  $X$ . But this is a contradiction as  $J_s$  is not countable. Therefore  $C_s \neq \emptyset$  for any  $s \in S$ . By Theorem 3.11 for each  $s \in S$ , we have  $C_s = \mu(Y_s)$  for some  $Y_s \in \mathcal{T}_D(X)$ . Suppose that  $s, t \in S$  and  $s \neq t$ . Let  $K = J_s \setminus J_t$  and let  $P = \bigcup_{i \in K} X_i$ . Then since  $|K| = \alpha$ , we have  $A = P^* \setminus \sigma X \neq \emptyset$ . But since  $P \cap Q_t = \emptyset$ , we have  $P^* \cap C_t = \emptyset$ , which implies that  $A \cap C_t = \emptyset$ . Therefore since  $\emptyset \neq A \subseteq C_s$ , we have  $C_s \neq C_t$ . Thus for any distinct  $s, t \in S$ , we have  $Y_s \neq Y_t$ . This shows that  $|\mathcal{T}_D(X)| \geq |S| = 2^\alpha$ , which together with the fact that  $\alpha = |I| \geq L(X)$  proves the theorem.  $\square$

## 7. SOME APPLICATIONS

In this section we correspond to each one-point extension of a Tychonoff space  $X$  an ideal of  $C^*(X)$ . Using this, and applying some of our previous results, we will be able to obtain some relations between the order structure of certain collections of ideals of  $C^*(X)$ , partially ordered by set-theoretic inclusion, and the topology of a certain subspace of  $X^*$ .

For a Tychonoff space  $X$ , let  $\mathcal{I}(C^*(X))$  denote the set of all ideals of  $C^*(X)$ . We define a function

$$\gamma : (\mathcal{T}(X), \leq) \rightarrow (\mathcal{I}(C^*(X)), \subseteq)$$

by

$$\gamma(Y) = \{f|X : f \in C^*(Y) \text{ and } f(p) = 0\}$$

for  $Y = X \cup \{p\} \in \mathcal{T}(X)$ .

**Lemma 7.1.** *The function  $\gamma$  is an order-isomorphism onto its image.*

*Proof.* To show that  $\gamma$  is well-defined, consider the functions  $g \in \gamma(Y)$  and  $h \in C^*(X)$ . Let  $f : Y \rightarrow \mathbf{R}$  be defined such that  $f(p) = 0$  and  $f|X = g.h$ . We verify that  $f$  is continuous. So let  $G \in C^*(Y)$  be such that  $G(p) = 0$  and  $G|X = g$ . Suppose that  $\epsilon > 0$ . Let  $W$  be an open neighborhood of  $p$  in  $Y$  such that  $G(W) \subseteq (-\epsilon/M, \epsilon/M)$ , where  $M > 0$  and  $|h(x)| \leq M$  for every  $x \in X$ . Then for every  $x \in W \cap X$  we have  $|f(x)| = |g(x)| < \epsilon$ . So  $f$  is continuous. Now  $g.h = f|X \in \gamma(Y)$ . It is clear that for any  $k, l \in \gamma(Y)$ ,  $k - l \in \gamma(Y)$ . This shows that  $\gamma$  is well-defined.

Now suppose that  $Y_i = X \cup \{p_i\} \in \mathcal{T}(X)$ , for  $i = 1, 2$ . Suppose that  $Y_1 \geq Y_2$  and let  $\phi : Y_1 \rightarrow Y_2$  be a continuous function such that  $\phi|X = \text{id}_X$ . Let  $g \in \gamma(Y_2)$ . Then  $g = f|X$ , where  $f \in C^*(Y_2)$  and  $f(p_2) = 0$ . Now since  $\phi(p_1) = p_2$ , we have  $g = f|X = f \circ \phi|X \in \gamma(Y_1)$ , i.e.,  $\gamma(Y_1) \supseteq \gamma(Y_2)$ .

Conversely, suppose that  $\gamma(Y_1) \supseteq \gamma(Y_2)$ . Define a function  $\phi : Y_1 \rightarrow Y_2$  by  $\phi|X = \text{id}_X$  and  $\phi(p_1) = p_2$ . To show that  $\phi$  is continuous at  $p_1$ , suppose that  $V$  is an open neighborhood of  $p_2 = \phi(p_1)$  in  $Y_2$ . Let  $f : Y_2 \rightarrow \mathbf{I}$  be a continuous function such that  $f(p_2) = 0$  and  $f(Y_2 \setminus V) \subseteq \{1\}$ . Then since  $f|X \in \gamma(Y_2)$ , we have  $f|X \in \gamma(Y_1)$  and therefore  $f|X = h|X$ , for some  $h \in C^*(Y_1)$  with  $h(p_1) = 0$ . Now  $U = h^{-1}([0, 1))$  is an open neighborhood of  $p_1$  in  $Y_1$  satisfying  $\phi(U) \subseteq V$ . This proves the continuity of  $\phi$  and therefore we have  $Y_1 \geq Y_2$ .  $\square$

The following result is well known. We include a proof in here for the sake of completeness.

**Lemma 7.2.** *Let  $X$  be a strongly zero-dimensional locally compact space. Then the set of clopen subset of  $X^*$  consist of exactly those sets which are of the form  $U^*$ , for some clopen subset  $U$  of  $X$ .*

*Proof.* Clearly for every clopen subset  $U$  of  $X$ , the set  $U^*$  is clopen in  $X^*$ . To see the converse suppose that  $C$  is a clopen subset of  $X^*$ . Let  $W$  be an open set of  $\beta X$  such that  $C = W \setminus X$ . Since  $C \subseteq X$  is compact, there exists a clopen subset  $V$  of  $\beta X$  such that  $C \subseteq V \subseteq W$ , and therefore

$$C = \text{cl}_{\beta X} V \setminus X = \text{cl}_{\beta X} (V \cap X) \setminus X = (V \cap X)^*.$$

□

For a Tychonoff space  $X$  and  $E \subseteq X$ , we let

$$I_E = \{g \in C^*(X) : |g|^{-1}([\epsilon, \infty)) \setminus E \text{ is compact for any } \epsilon > 0\}.$$

It is easy to see that if  $E$  is open in  $X$  then  $I_E$  is an ideal in  $C^*(X)$ .

**Lemma 7.3.** *Let  $X$  be a locally compact space and let  $U$  be a clopen subset of  $X$ . If  $Y \in \mathcal{T}(X)$  is such that  $\mu(Y) = X^* \setminus U^*$  then  $\gamma(Y) = I_U$ .*

*Proof.* Suppose that  $g \in \gamma(Y)$ . Then  $g = f|X$  for some  $f \in C^*(Y)$  with  $f(p) = 0$ . Suppose that there exists an  $\epsilon > 0$  such that  $G = |g|^{-1}([\epsilon, \infty)) \setminus U$  is not compact, and let  $x \in G^*$ . By continuity of  $f$  there exists an open neighborhood  $W$  of  $p$  in  $\beta Y$  such that  $f(W \cap Y) \subseteq (-\epsilon, \epsilon)$ . Since  $p \in W$ ,  $X^* \setminus U^* = q^{-1}(p) \subseteq q^{-1}(W)$ , where  $q : \beta X \rightarrow \beta Y$  is the quotient map contracting  $X^* \setminus U^*$  to the point  $p$ . Now since  $x \in \text{cl}_{\beta X} G \subseteq \text{cl}_{\beta X} (X \setminus U)$  and  $U$  is clopen in  $X$ , where  $x \notin \text{cl}_{\beta X} U$  and thus  $x \in X^* \setminus U^* \subseteq q^{-1}(W)$ , which implies that  $G \cap q^{-1}(W) \neq \emptyset$ . Let  $t \in G \cap q^{-1}(W)$ . Then since  $t \in G$ , we have  $|f(t)| = |g(t)| \geq \epsilon$ . But on the other hand,  $t = q(t) \in W$  and by the way we chose  $W$ , we have  $|f(t)| < \epsilon$ . This is a contradiction, therefore  $|g|^{-1}([\epsilon, \infty)) \setminus U$  is compact for any  $\epsilon > 0$ , and thus  $g \in I_U$ . This shows that  $\gamma(Y) \subseteq I_U$ .

Conversely, let  $g \in I_U$  and define a function  $f : Y \rightarrow \mathbf{R}$  such that  $f|X = g$  and  $f(p) = 0$ . We verify that  $f$  is continuous. So suppose that  $\epsilon > 0$ . Since  $U$  is clopen in  $X$ , the set  $X^* \setminus U^*$  is compact and it is disjoint from the compact subset  $G = |g|^{-1}([\epsilon, \infty)) \setminus U$  of  $X$ . Let  $W$  be an open neighborhood of  $X^* \setminus U^*$  in  $\beta X$  disjoint from  $G$ . Then  $V = W \setminus \text{cl}_{\beta X} U$  is an open set of  $\beta X$  containing  $X^* \setminus U^*$ . Clearly  $T = (V \setminus (X^* \setminus U^*)) \cup \{p\}$  is open in  $\beta Y$ . Now  $T \cap Y$  is an open neighborhood of  $p$  in  $Y$  such that  $f(T \cap Y) \subseteq (-\epsilon, \epsilon)$ . This is because if  $t \in T \cap X$ , then since  $T \cap X \subseteq V$  and  $W \cap G = \emptyset$ , we have  $t \notin G$  and  $t \notin \text{cl}_{\beta X} U$ , and therefore  $t \notin |g|^{-1}([\epsilon, \infty))$ . Thus  $|f(t)| = |g(t)| < \epsilon$ . This shows that  $f$  is continuous and therefore  $g = f|X \in \gamma(Y)$ , i.e.,  $I_U \subseteq \gamma(Y)$ . □

For a Tychonoff space  $X$ , let

$$\Sigma_X = \{I_U : U \text{ is a } \sigma\text{-compact clopen subset of } X\}.$$

**Lemma 7.4.** *Let  $X$  be a zero-dimensional locally compact paracompact non- $\sigma$ -compact space. Then*

$$\gamma(\mathcal{T}_{KL}(X)) = \Sigma_X.$$

*Proof.* Suppose that  $Y \in \mathcal{T}_{KL}(X)$  and let  $C = \mu(Y)$ . Then by Theorem 2.6 and Lemma 3.12 we have  $C$  is clopen in  $X^*$  and contains  $\beta X \setminus \sigma X$ . Now  $X$ , being zero-dimensional, locally compact and paracompact, is strongly zero-dimensional (see Theorem 6.2.10 of [4]) and thus by Lemma 7.2 we have  $X^* \setminus C = U^*$ , for some clopen  $U \subseteq X$ . By Lemma 7.3 we have  $\gamma(Y) = I_U$ . But since  $X^* \setminus U^* = C \supseteq \beta X \setminus \sigma X$ , we have  $\text{cl}_{\beta X} U \subseteq \sigma X$  and thus  $U$  is  $\sigma$ -compact. This shows that  $\gamma(Y) \in \Sigma_X$ , i.e.,  $\gamma(\mathcal{T}_{KL}(X)) \subseteq \Sigma_X$ .

Conversely, if  $U$  is a  $\sigma$ -compact clopen subset of  $X$ , then  $\text{cl}_{\beta X} U$  is clopen in  $\beta X$  and is contained in  $\sigma X$ , therefore  $C = X^* \setminus U^* \supseteq \beta X \setminus \sigma X$ . If we let  $\mu(Y) = C$ , then  $Y \in \mathcal{T}_{KL}(X)$ , and by Lemma 7.3 we have  $\gamma(Y) = I_U$ , which shows that  $\Sigma_X \subseteq \gamma(\mathcal{T}_{KL}(X))$ .  $\square$

For a Tychonoff space  $X$ , let

$$\Delta_X = \{I_{X \setminus U} : U \text{ is a } \sigma\text{-compact non-compact clopen subset of } X\}.$$

**Lemma 7.5.** *Let  $X$  be a zero-dimensional locally compact paracompact non- $\sigma$ -compact space. Then*

$$\gamma(\mathcal{T}_K^*(X)) = \Delta_X.$$

*Proof.* Suppose that  $Y \in \mathcal{T}_K^*(X)$ . By Lemma 3.2 the set  $\mu(Y)$  is clopen in  $X^*$ . Now  $X$  being zero-dimensional, locally compact and paracompact, is strongly zero-dimensional (see Theorem 6.2.10 of [4]) therefore by Lemma 7.2 we have  $\mu(Y) = U^*$ , for some clopen subset  $U$  of  $X$ . By Lemma 3.8 we have  $U^* = \mu(Y) \subseteq \sigma X$ , which implies that  $U$  is  $\sigma$ -compact and thus by Lemma 7.3, we have  $\gamma(Y) = I_{X \setminus U} \in \Delta_X$ .

For the converse, let  $U$  be a  $\sigma$ -compact non-compact clopen subset of  $X$ . Then by Lemma 16 we have  $U^* = \mu(Y)$ , for some  $Y \in \mathcal{T}_K^*(X)$ . Now since  $\mu(Y) = X^* \setminus (X \setminus U)^*$ , by Lemma 7.3, we have  $I_{X \setminus U} = \mu(Y) \in \gamma(\mathcal{T}_K^*(X))$ .  $\square$

**Theorem 7.6.** *For zero-dimensional locally compact paracompact non- $\sigma$ -compact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $(\Sigma_X, \subseteq)$  and  $(\Sigma_Y, \subseteq)$  are order-isomorphic;
- (2)  $(\Delta_X, \subseteq)$  and  $(\Delta_Y, \subseteq)$  are order-isomorphic;
- (3)  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

*Proof.* This follows from Lemmas 7.1, 7.4 and 7.5, Corollary 4.12 and Theorem 3.9.  $\square$

**Definition 7.7.** Let  $X$  be a Tychonoff space. A sequence  $\{U_n\}_{n < \omega}$  is called a  $\sigma$ -regular sequence of open sets in  $X$ , if for each  $n < \omega$ , the set  $U_n$  is open in  $X$  and is such that  $\text{cl}_X U_n$  is  $\sigma$ -compact and non-compact, and  $U_n \supseteq \text{cl}_X U_{n+1}$ .

If  $\mathcal{U} = \{U_n\}_{n < \omega}$  is a  $\sigma$ -regular sequence of open sets in  $X$ , we let  $I_{\mathcal{U}}$  denote the set

$$\{g \in C^*(X) : \text{for any } \epsilon > 0, |g|^{-1}([\epsilon, \infty)) \cap \text{cl}_X U_n \text{ is compact for some } n < \omega\}$$

and let

$$\Omega_X = \{I_{\mathcal{U}} : \mathcal{U} \text{ is a } \sigma\text{-regular sequence of open sets in } X\}.$$

**Lemma 7.8.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space and let  $\mathcal{U} = \{U_n\}_{n < \omega}$  be a  $\sigma$ -regular sequence of open sets in  $X$ . Suppose that  $\{f_n\}_{n < \omega}$  is a sequence in  $C(\beta X, \mathbf{I})$  such that for each  $n < \omega$ , we have  $f_n(U_{n+1}) \subseteq \{0\}$  and  $f_n(X \setminus U_n) \subseteq \{1\}$ . Then  $C = \bigcap_{n < \omega} Z(f_n) \setminus X \in \mu(\mathcal{T}(X))$ , and if  $Y \in \mathcal{T}(X)$  is such that  $\mu(Y) = C$ , then we have  $\gamma(Y) = I_{\mathcal{U}}$ .*

*Proof.* First note that since for each  $n < \omega$ , we have  $\text{cl}_{\beta X} U_{n+1} \subseteq Z(f_n)$ , and  $\emptyset \neq \bigcap_{n < \omega} U_n^* \subseteq C$  and so  $C \in \mu(\mathcal{T}(X))$ . Suppose that  $C = \mu(Y)$ , for some  $Y = X \cup \{p\} \in \mathcal{T}(X)$ . Let  $g \in \gamma(Y)$ . Then  $g = f|_X$  for some  $f \in C^*(Y)$  with  $f(p) = 0$ . Suppose that  $g \notin I_{\mathcal{U}}$ . Then there exists an  $\epsilon > 0$  such that for each  $n < \omega$ , the set  $A_n = |g|^{-1}([\epsilon, \infty)) \cap \text{cl}_X U_n$  is not compact. By compactness of  $X^*$  we have  $\bigcap_{n < \omega} A_n^* \neq \emptyset$ . Let  $x \in \bigcap_{n < \omega} A_n^*$ . Then  $x \in C$ . Let  $Z$  be the space



obtained from  $\beta X$  by contracting  $C$  to the point  $p$ , and let  $q : \beta X \rightarrow Z = \beta Y$  be its natural quotient mapping. By continuity of  $f$  there exists an open neighborhood  $V$  of  $p$  in  $Y$  such that  $f(V) \subseteq (-\epsilon, \epsilon)$ . Let  $W$  be an open subset of  $\beta Y$  with  $W \cap Y = V$ . Then since  $p \in W$ , the set  $C \subseteq q^{-1}(W)$ . Now since  $\bigcap_{n < \omega} A_n^* \subseteq C$ , the set  $q^{-1}(W)$  is an open neighborhood of  $x$  in  $\beta X$ , and therefore since  $x \in \text{cl}_{\beta X} A_1$ , we have  $A_1 \cap q^{-1}(W) \neq \emptyset$ . Let  $t \in A_1 \cap q^{-1}(W)$ . Then  $t = q(t) \in W$  and thus  $|g(t)| = |f(t)| < \epsilon$ . But since  $t \in A_1$ , we have  $|g(t)| \geq \epsilon$ , which is a contradiction. This shows that  $g \in I_{\mathcal{U}}$ . Thus  $\gamma(Y) \subseteq I_{\mathcal{U}}$ .

To show the reverse inclusion, let  $g \in I_{\mathcal{U}}$ . Define a function  $f : Y \rightarrow \mathbf{R}$  such that  $f|_X = g$  and  $f(p) = 0$ . We show that  $f$  is continuous at  $p$ . So let  $\epsilon > 0$ . Then by assumptions there exists a  $k < \omega$  such that  $S = |g|^{-1}([\epsilon, \infty)) \cap \text{cl}_X U_k$  is compact. Since  $C \subseteq f_k^{-1}([0, 1))$ , the set  $T = (f_k^{-1}([0, 1)) \setminus C) \cup \{p\}$  is open in  $\beta Y$ . Consider the open neighborhood  $(T \cap Y) \setminus S$  of  $p$  in  $Y$ . If  $t \in (T \cap Y) \setminus S$ , then since  $t \in T$ , we have  $f_k(t) < 1$  and so  $t \in U_k$ . But  $t \notin S$  and therefore  $|f(t)| = |g(t)| < \epsilon$ , i.e.,  $f((T \cap Y) \setminus S) \subseteq (-\epsilon, \epsilon)$ . This shows the continuity of  $f$  and thus  $g \in \gamma(Y)$ . Therefore  $I_{\mathcal{U}} \subseteq \gamma(Y)$ , which together with the previous part of the proof proves the lemma.  $\square$

**Lemma 7.9.** *Let  $X$  be a locally compact paracompact non- $\sigma$ -compact space. Then*

$$\gamma(\mathcal{T}_S^*(X)) = \Omega_X.$$

*Proof.* Assume the notations of Proposition 3.1. Suppose that  $Y = X \cup \{p\} \in \mathcal{T}_S^*(X)$ . By Lemma 3.18, we have  $\mu(Y) = C \in \mathcal{Z}(\beta X)$  and  $C \subseteq \sigma X$ . Therefore  $C \subseteq \text{cl}_{\beta X} M$ , where  $M = \bigcup_{i \in J} X_i$ , for some countable  $J \subseteq I$ . Let  $h \in C(\beta X, \mathbf{I})$  be such that  $Z(h) = C$  and  $h(\beta X \setminus \text{cl}_{\beta X} M) \subseteq \{1\}$ . For each  $n < \omega$  let  $U_n = h^{-1}([0, 1/n)) \cap X$ . Then since for each  $n < \omega$ ,  $C \subseteq h^{-1}([0, 1/n))$ , we have  $U_n \neq \emptyset$ , and since  $U_n \subseteq M$  and  $C \subseteq \text{cl}_{\beta X} U_n$ ,  $\text{cl}_X U_n$  is  $\sigma$ -compact and non-compact. Clearly for each  $n < \omega$ , we have  $U_n \supseteq \text{cl}_X U_{n+1}$ , which shows that  $\mathcal{U} = \{U_n\}_{n < \omega}$  is a  $\sigma$ -regular sequence of open sets in  $X$ .

For each  $n < \omega$  define  $f_n : \beta X \rightarrow \mathbf{I}$  by

$$f_n = \left( \left( \left( h \wedge \frac{1}{n} \right) \vee \frac{1}{n+1} \right) - \frac{1}{n+1} \right) \left( \frac{1}{n} - \frac{1}{n+1} \right)^{-1}.$$

Then the sequence  $\{f_n\}_{n < \omega}$  satisfies the requirements of Lemma 7.8, and therefore since  $\mu(Y) = Z(h) = \bigcap_{n < \omega} Z(f_n)$ , we have  $\gamma(Y) = I_{\mathcal{U}}$ . This shows that  $\gamma(\mathcal{T}_S^*(X)) \subseteq \Omega_X$ .

To complete the proof we need to show that  $\Omega_X \subseteq \gamma(\mathcal{T}_S^*(X))$ . So let  $\mathcal{U} = \{U_n\}_{n < \omega}$  be a  $\sigma$ -regular sequence of open sets in  $X$ . We verify that  $I_{\mathcal{U}} \in \gamma(\mathcal{T}_S^*(X))$ . For each  $n < \omega$  since  $U_n \supseteq \text{cl}_X U_{n+1}$ , by normality of  $X$ , there exists an  $f_n \in C(X, \mathbf{I})$  such that  $f_n(\text{cl}_X U_{n+1}) \subseteq \{0\}$  and  $f_n(X \setminus U_n) \subseteq \{1\}$ . Let  $F_n \in C(\beta X, \mathbf{I})$  be the extension of  $f_n$ . Since each  $\text{cl}_X U_{n+1}$  is  $\sigma$ -compact, for each  $n < \omega$  we have  $U_n \subseteq P$ , where  $P = \bigcup_{i \in L} X_i$  and  $L \subseteq I$  is countable. Since  $X \setminus P \subseteq X \setminus U_n \subseteq F_n^{-1}(1)$ , it follows that  $Z(F_n) \subseteq \beta X \setminus \text{cl}_{\beta X}(X \setminus P) = \text{cl}_{\beta X} P$  and therefore since  $P^* \in \mathcal{Z}(\beta X)$  we have  $D = \bigcap_{n < \omega} Z(F_n) \setminus X \in \mathcal{Z}(\beta X)$ . But  $\emptyset \neq U_n^* \subseteq Z(F_n)$  and  $D \subseteq \sigma X$ , which by Lemma 3.18 implies that  $D = \mu(Y)$ , for some  $Y = X \cup \{p\} \in \mathcal{T}_S^*(X)$ . Now the sequence  $\{F_n\}_{n < \omega}$  satisfies the requirements of Lemma 7.8 and therefore  $\gamma(Y) = I_{\mathcal{U}}$ . This shows that  $\Omega_X \subseteq \gamma(\mathcal{T}_S^*(X))$ , which together with the first part of the proof give the result.  $\square$

Now from Theorem 3.21 and the above lemmas we obtain the following result.

**Theorem 7.10.** *For locally compact paracompact non- $\sigma$ -compact spaces  $X$  and  $Y$  the following conditions are equivalent.*

- (1)  $(\Omega_X, \subseteq)$  and  $(\Omega_Y, \subseteq)$  are order-isomorphic;
- (2)  $\sigma X \setminus X$  and  $\sigma Y \setminus Y$  are homeomorphic.

**Acknowledgments.** The author would like to express his deep gratitude to Professor R. Grant Woods for his invaluable comments during this work. The author also thanks the referee for reading the manuscript and his comments.

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DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY, ISFAHAN 84156–83111, IRAN

*E-mail address:* koushesh@cc.iut.ac.ir